MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

Models of uninhibited growth

The British economist Thomas R. Malthus (1766–1834) proposed a model of natural population growth that assumes that both the birth rate and the death rate of a population are proportional to the current size of the population at each instant.

Consequently, the net rate of change in population can be restated as:
'net rate of change in population = birth rate − death rate'.

In other words: Let $P(t)$ represent the size of a population at time t. Assuming that the population level is specified at the initial time $t = 0$ as $P(0) = P_0$, predict the population at later times t.

$$\frac{dP}{dt} = bP - dP$$

$$\frac{dP}{dt} = (b - d)P$$

$$\frac{dP}{dt} = rP, \text{ where } r = b - d \text{ with } P(0) = P_0.$$

In summary, Malthus' uninhibited growth model $\frac{1}{P}\frac{dP}{dt} = r$ has a constant relative growth rate r. This model results in exponential growth for P. In more general terms:

A dependent variable $y(t)$ is said to have an *exponential growth rate* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = k > 0$, or an *exponential rate of decay* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = -k < 0$ for some positive constant k.

Finding the doubling-time in a model of exponential growth

Example 21

Consider the differential equation $\frac{dy}{dt} = ky$, given that $y(0) = y_0 > 0$, with $k > 0$.
Find the time T for y to double its initial value y_0.

Solution

Rearrange to separate the two variables:

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\frac{dy}{dt} = k$$

Integrate with respect to the independent variable:

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt$$

$$\log_e|y| = kt + c$$

Exponentiate:

$$|y| = e^{kt+c}$$

The general solution is: $y = Ae^{kt}$ where $A = \pm e^c$

Apply the initial condition $y(0) = y_0$ to solve for the constant of integration: $y_0 = A$

Substitute to find the model of exponential growth: $y = y_0 e^{kt}$

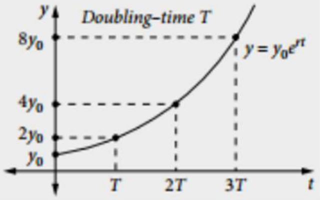
At time T, $y = 2y_0$:

$$2y_0 = y_0 e^{kT}$$

$$2 = e^{kT}$$

$$T = \frac{1}{k}\log_e 2$$

For a system $y = y_0 e^{rt}$, $r > 0$ experiencing exponential growth, the doubling-time (T) is the time it takes the system to double in size. The formula for the doubling-time is $T = \frac{1}{r}\log_e 2 = \frac{(t_2 - t_1)}{\log_e\left(\frac{y(t_1)}{y(t_2)}\right)}\log_e 2$



Radioactive decay

Example 22

The New Zealand-born physicist Ernest Rutherford is often called the father of nuclear physics. Rutherford was able to formulate a model of radioactive decay as a simple differential equation. From his laboratory experiments Rutherford discovered that the relative rate r, at which atoms of radioactive materials disintegrate, is proportional to the number of atoms N still present in the sample. In other words, this model states that $\frac{dN}{dt} = -rN,\ N(0) = N_0$.

(a) Show that the solution of this model is $N = N_0 e^{-rt}$.

(b) Hence find the time $t_{\frac{1}{2}}$ required for the number of atoms still present (N) to fall to $\frac{1}{2}N_0$.

Solution

(a) The two variables are separated: $\frac{1}{N}\frac{dN}{dt} = -r$

Both sides of the equation are integrated with respect to the independent variable:

$$\int \frac{1}{N}\frac{dN}{dt}\,dt = -\int r\,dt$$

$$\log_e|N| = -rt + c$$

$$N = \pm e^{-rt+c}$$

$$N = \pm e^{c} e^{-rt}$$

$$\therefore \quad N = Ae^{-rt} \quad \text{where } A = \pm e^{c}$$

The initial conditions are substituted in the last equation to find the constant of integration, $N(0) = N_0$, $N_0 > 0$:

$A = \pm e^{c} = N_0$

$\therefore\ N = N_0 e^{-rt}$ as required.

(b) The conditions required are substituted into the equation from part **(a)**: $N = N_0 e^{-rt}$ with $N = \frac{1}{2}N_0$

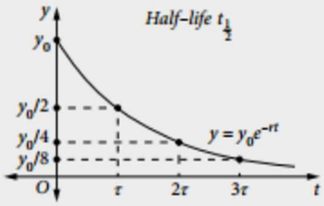
$$\frac{1}{2}N_0 = N_0 e^{-rt}$$

$$\frac{1}{2} = e^{-rt}$$

The exponential equations are changed into a logarithmic form: $t = \frac{1}{r}\log_e 2$.

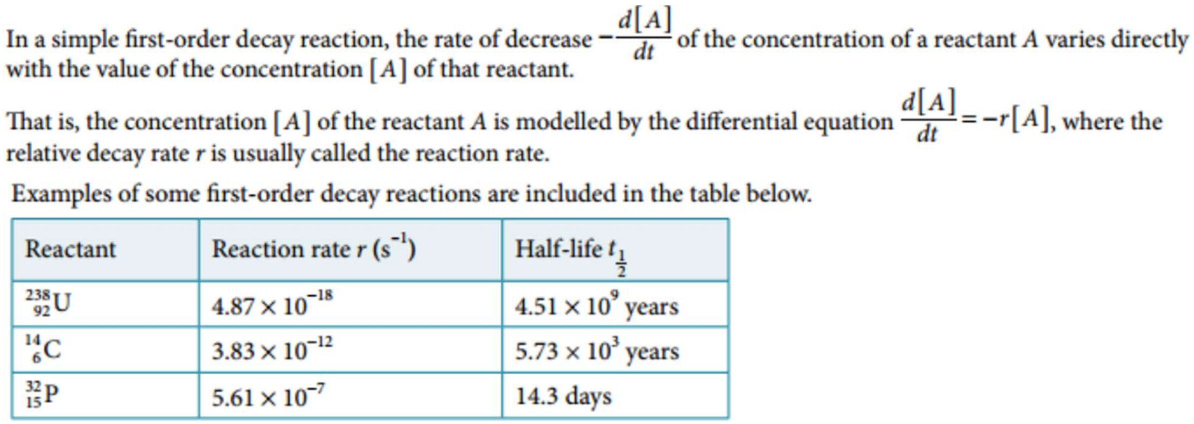
Hence $t_{\frac{1}{2}} = \frac{1}{r}\log_e 2$, where $t_{\frac{1}{2}}$ is the half-life of the radioactive material.

For a system $y = y_0 e^{-rt}$, $r > 0$ experiencing exponential decay, the half-life, $t_{\frac{1}{2}}$ is the time it takes the value of y to halve. The formula for the half-life is $t_{\frac{1}{2}} = \frac{1}{r}\log_e 2 = \dfrac{(t_2 - t_1)}{\log_e\left(\dfrac{y(t_1)}{y(t_2)}\right)}\log_e 2$.



MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

First-order decay reactions



In a simple first-order decay reaction, the rate of decrease $-\frac{d[A]}{dt}$ of the concentration of a reactant A varies directly with the value of the concentration $[A]$ of that reactant.

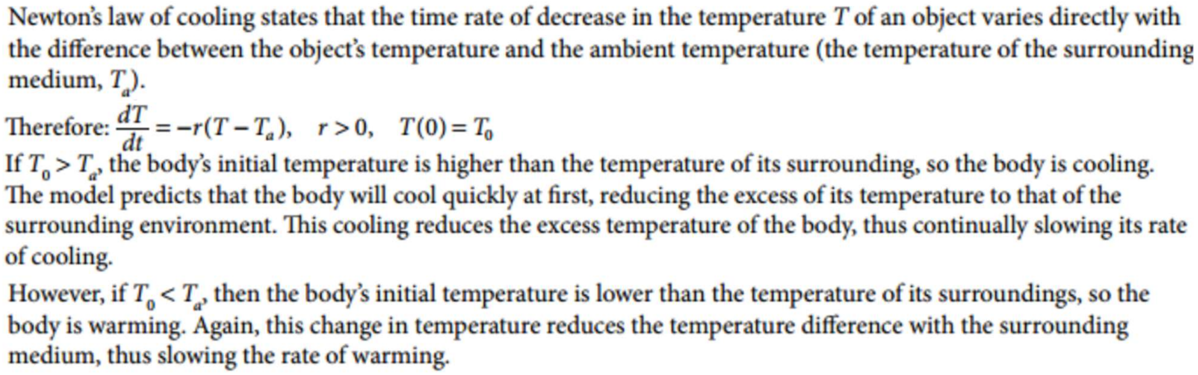
That is, the concentration $[A]$ of the reactant A is modelled by the differential equation $\frac{d[A]}{dt} = -r[A]$, where the relative decay rate r is usually called the reaction rate.

Examples of some first-order decay reactions are included in the table below.

Reactant	Reaction rate r (s^{-1})	Half-life $t_{\frac{1}{2}}$
$^{238}_{92}U$	4.87×10^{-18}	4.51×10^{9} years
$^{14}_{6}C$	3.83×10^{-12}	5.73×10^{3} years
$^{32}_{15}P$	5.61×10^{-7}	14.3 days

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

Newton's law of cooling



Newton's law of cooling states that the time rate of decrease in the temperature T of an object varies directly with the difference between the object's temperature and the ambient temperature (the temperature of the surrounding medium, T_a).

Therefore: $\frac{dT}{dt} = -r(T - T_a), \quad r > 0, \quad T(0) = T_0$

If $T_0 > T_a$, the body's initial temperature is higher than the temperature of its surrounding, so the body is cooling. The model predicts that the body will cool quickly at first, reducing the excess of its temperature to that of the surrounding environment. This cooling reduces the excess temperature of the body, thus continually slowing its rate of cooling.

However, if $T_0 < T_a$, then the body's initial temperature is lower than the temperature of its surroundings, so the body is warming. Again, this change in temperature reduces the temperature difference with the surrounding medium, thus slowing the rate of warming.

Example 23

Newton's law of cooling can be used to model the temperature of a cup of coffee cooling on a kitchen bench. If the temperature of the kitchen is a constant 20°C and the initial temperature of the coffee is 95°C, after 20 minutes the coffee will have cooled to 65°C.

(a) Find the temperature of the coffee after an additional 20 minutes.

(b) Sketch a graph of the temperature of the coffee for the first 2 hours.

Solution

(a) This problem models the process of cooling, so Newton's law of cooling is: $\frac{dT}{dt} = -r(T-20),\ r > 0$, $T(0) = 95$.

The two variables are separated: $\frac{1}{(T-20)}\frac{dT}{dt} = -r$

Both sides of the equation are integrated with respect to t and a change of variable is applied on the LHS of the equation: $\int \frac{1}{(T-20)}\frac{dT}{dt}\,dt = -\int r\,dt$

$$\int \frac{1}{(T-20)}\,dT = -\int r\,dt$$

$$\log_e|T-20| = -rt + c$$

Both sides of the equation are exponentiated and solved for the dependent variable: $T - 20 = Ae^{-rt}$, where $A = \pm e^c$.

The initial condition is substituted and the constant of integration is determined:

$T(0) = 95$: $95 - 20 = A$

$\therefore\ T = 20 + 75e^{-rt}$

$t = 20$, $T = 65$, is substituted to determine the constant of proportionality: $\log_e\left(\frac{45}{75}\right) = -20r$

$$r = \frac{1}{20}\log_e\left(\frac{5}{3}\right)$$

The constant of proportionality is substituted in the general solution: $T = 20 + 75e^{-\frac{t}{20}\log_e\left(\frac{5}{3}\right)}$

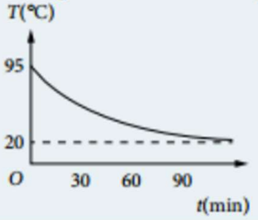
Now: $e^{-\frac{t}{20}\log_e\left(\frac{5}{3}\right)} = e^{\frac{t}{20}\log_e\left(\frac{3}{5}\right)} = e^{\log_e\left(\frac{3}{5}\right)^{\frac{t}{20}}} = \left(\frac{3}{5}\right)^{\frac{t}{20}}$

Simplify: $T(t) = 20 + 75\left(\frac{3}{5}\right)^{\frac{t}{20}}$

After an additional 20 minutes, $t = 40$: $T(40) = 20 + 75\left(\frac{3}{5}\right)^2 = 20 + 27 = 47°\text{C}$

The temperature of the coffee is 47°C.

(b) Using the model from part **(a)** to plot the changing value of T:



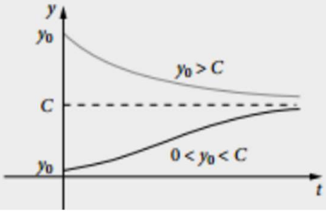
MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

Models of modified growth and decay

The uninhibited growth model with its constant relative growth rate $r = \frac{1}{y}\frac{dy}{dt}$ does not take into account the inherent limitations on the growth of a population. In practice, most populations have a size (the carrying capacity C) beyond which their environment can no longer sustain them. To account for the finite carrying capacity inherent in most systems, the Belgian mathematician Pierre François Verhulst (1804–1849) modified the uninhibited growth model by replacing constant relative growth rate r with a damped growth rate $r\left(1 - \frac{y}{C}\right)$.

Verhulst's modified growth rate model is then equivalent to $\frac{1}{y}\frac{dy}{dt} = r\left(1 - \frac{y}{C}\right)$, which slows the rate of growth as the population approaches the 'carrying capacity' $y = C$ of the system.

The differential equation $\frac{dy}{dt} = ry\left(1 - \frac{y}{C}\right)$, $y(0) = y_0$ where $r > 0$ is called a **logistic equation**. This models population growth in an environment with a finite carrying capacity $C > 0$.



Example 24

Rabbit Island currently has 5000 rabbits, but has sufficient space and food for 20 000 rabbits in total.

The model describing this rabbit population P is then $\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{20000}\right)$.

Find the particular solution of this differential equation, given that $\frac{20000}{P(20000 - P)} = \frac{1}{P} + \frac{1}{20000 - P}$.

Solution

Take the reciprocal of both sides of the differential equation: $\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{20000}\right)$

$$\frac{dt}{dP} = \frac{200\,000}{P(20000 - P)}$$

Integrate both sides of the equation with respect to the original dependent variable:

Given that $\frac{200\,000}{P(20000 - P)} = 10\left(\frac{1}{P} + \frac{1}{20000 - P}\right)$

$$\int \frac{dt}{dP}\,dP = \int \frac{200\,000}{P(20000 - P)}\,dP$$

$$\frac{1}{10}\int dt = \int \left(\frac{1}{P} + \frac{1}{20\,000 - P}\right)dP$$

$$\frac{t}{10} + c = \log_e\left|\frac{P}{20\,000 - P}\right|$$

Find the general solution: $Ae^{\frac{t}{10}} = \frac{P}{20000 - P}$, where $A = \pm e^c$

Substitute the initial conditions $t = 0$, $P = 5000$: $A = \frac{5000}{20000 - 5000} = \frac{1}{3}$

$$P = \frac{20000e^{\frac{t}{10}}}{3 + e^{\frac{t}{10}}}$$

$$= \frac{20\,000}{1 + 3e^{\frac{-t}{10}}}$$

MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

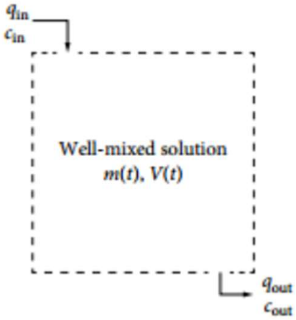
Modelling the time dependence of a conserved substance

Differential equations can be used to model the time behaviour of a conserved quantity within a confined region of space, if you know the rate at which the quantity flows into and out of the region. Such models are of great practical value in the fields of Science and Engineering.

If a substance m is conserved (that is, neither created nor destroyed), then the rate of change of m within a confined region (called a *control volume*) is equal to the difference between the rate at which it flows into the region (R_{in}), and the rate at which it flows out of the region (R_{out}). That is, $\frac{dm}{dt} = R_{\text{in}} - R_{out}$.

Consider the situation shown at right:

- $m(t)$ is the mass of some conserved substance (e.g. salt or sugar) in the control volume at time t
- $V(t)$ is the volume of the well-mixed solution within the control volume at time t
- q_{in}, measured in units of $\left(\frac{volume}{time}\right)$, is the rate at which the solution flows into the control volume
- q_{out}, measured in units of $\frac{volume}{time}$, is the rate at which the solution flows out of the control volume
- c_{in}, measured in units of $\left(\frac{mass}{volume}\right)$, is the concentration of the conserved substance in the inflow
- c_{out}, measured in units of $\left(\frac{mass}{volume}\right)$, is the concentration of the conserved substance in the outflow



The rate R_{in} at which the conserved substance $m(t)$ flows into the control volume is measured in units of $\frac{mass}{time}$ so that $R_{in}\left[\frac{mass}{time}\right] = q_{in}\left[\frac{volume}{time}\right] \times c_{in}\left[\frac{mass}{volume}\right]$

Similarly, the rate at which $m(t)$ flows out of the control volume is $R_{out}\left[\frac{mass}{time}\right] = q_{out}\left[\frac{volume}{time}\right] \times c_{out}\left[\frac{mass}{volume}\right]$

Assume that $m(t)$ is conserved, so it is neither created nor destroyed within the control volume. Therefore $\frac{dm}{dt} = R_{in} - R_{out}$.

This means that the net rate of change in the mass of the solute $m(t)$ within the control volume at time t, is modelled by the solution of differential equation $\frac{dm}{dt} = q_{in} \times c_{in} - q_{out} \times c_{out}$.

To solve this equation, the mass concentration of the solute in the outflow c_{out} must be expressed in terms of the dependent variable $m(t)$ and the control volume $V(t)$ as $c_{out} = \frac{m(t)}{V(t)}$.

However, the net rate of change of the volume of the solution within the control volume is modelled by the simple differential equation $\frac{dV}{dt} = q_{in} - q_{out}$, which gives the solution $V(t) = V(0) + (q_{in} - q_{out})t$.

The differential equation for the time rate of change of the conserved substance within the control volume can now be written as $\frac{dm}{dt} = q_{in} \times c_{in} - q_{out} \times \frac{m(t)}{V(0) + (q_{in} - q_{out})t}$.

Modelling the mass of a conserved quantity

Example 25

A holding tank at a desalination plant contains 200 cubic metres of brine solution (salt in water), 20 tonnes of which is salt. At time $t = 0$ minutes, seawater starts to flow into this at a rate of 10 cubic metres per minute. This seawater has a salt concentration of 35 kilograms per cubic metre. The 'well-mixed' solution (brine mixed with seawater) also flows out of the holding tank at the same rate of 10 cubic metres per minute.

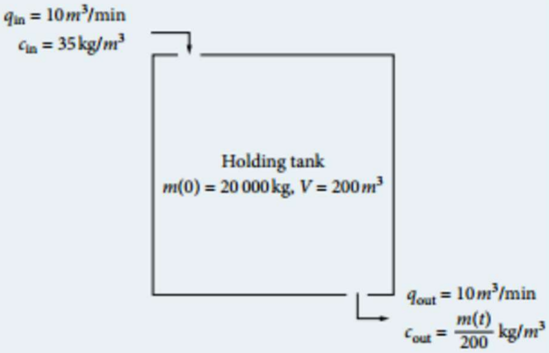
(a) Construct a differential equation to model the mass m kg of salt in the tank after t minutes.

(b) State the amount of salt within the holding tank $m(t)$ kg after t minutes.

(c) As the water continues to flow through the tank, over time the concentration of salt within the tank approaches a 'long-term' value. What is this value, in units of kg/m^3?

(d) Sketch the time dependence of the salt concentration of the 'well-mixed' outflow.

Solution



(a) Volume of the solution is fixed at 200 m^3, because $q_{in} = q_{out} = 10$ m^3/min.

Concentration of the inflow $c_{in} = 35$ kg per m^3.

Concentration of the outflow

$c_{out} = \frac{m(t)}{200}$ kg per m^3, where $m(t)$ is the kilograms of salt within the tank after t minutes.

Initial mass of salt is $m(0) = 20\,000$ kg.

Net rate of change of salt $\left(\frac{\text{kg}}{\text{min}}\right)$ = rate of inflow $\left(\frac{\text{kg}}{\text{min}}\right)$ − rate of outflow $\left(\frac{\text{kg}}{\text{min}}\right)$

$$\frac{dm}{dt} = 350 - \frac{m}{20}$$

$$= \frac{1}{20}(7\,000 - m) \text{ given that } m(0) = 20\,000 \text{ kg}$$

(b) The model is an example of the general inhibited decay model:

$\frac{dy}{dt} = r(a - y), y(0) = y_0 \therefore y = a + (y_0 - a)e^{-rt}$

and thus, has the solution $m(t) = 7000 + 13000e^{\frac{-t}{20}}$

(c) Any decaying exponentials will vanish over time:

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} 1000\left(7 + 13e^{\frac{-t}{20}}\right) = 7000 \text{ kg}$$

(d) The concentration c_{out} of the outflow is the mass divided by the volume of the tank: $c_{out} = \frac{m}{200}$ kg/m^3.

The mass exhibits an inhibited decay towards its equilibrium value of 7000 kg. This corresponds to a mass concentration of 35 kg/m^3.