# PARTIAL FRACTIONS, QUADRATIC FACTORS

Consider the case when $B(x)$ is a product of distinct linear factors and a simple quadratic factor, so that the decomposition is of the form: $\frac{R(x)}{B(x)} = \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \ldots + \frac{c_n}{x-a_n} + \frac{dx+e}{x^2+bx+c}$, with degree of $R(x) < n+2$ (as the numerator has a degree at least one less than the denominator).

As for partial fractions with linear factors, there are many different methods that may be used.

## Example 8

Reduce $\frac{13}{(x-3)(x^2+4)}$ to its partial fractions.

### Solution

Let $\frac{13}{(x-3)(x^2+4)} = \frac{c}{x-3} + \frac{dx+e}{x^2+4}$

$R(x) = 13 \qquad B(x) = x^3 - 3x^2 + 4x - 12 \qquad B'(x) = 3x^2 - 6x + 4$

For $c$, let $x = 3$: $\quad c = \frac{R(3)}{B'(3)} = \frac{13}{27-18+4} = 1$

For $e$, let $x = 0$: $\quad \frac{13}{(-3)(4)} = \frac{1}{(-3)} + \frac{e}{4} \quad \therefore e = -3$

Hence: $\quad \frac{13}{(x-3)(x^2+4)} = \frac{1}{x-3} + \frac{dx-3}{x^2+4}$

For $d$, multiply through by $x$: $\quad \frac{13x}{(x-3)(x^2+4)} = \frac{x}{x-3} + \frac{dx^2-3x}{x^2+4}$

Divide by largest power of $x$ in each numerator: $\quad \frac{13}{\left(1-\frac{3}{x}\right)\left(x^2+4\right)} = \frac{1}{1-\frac{3}{x}} + \frac{d-\frac{3}{x}}{1+\frac{4}{x^2}}$

Find $\lim_{x\to\infty}$ of both sides: $\quad \frac{13}{\infty} = \frac{1}{1} + \frac{d}{1}$

$$0 = 1 + d$$
$$d = -1$$

Thus: $\frac{13}{(x-3)(x^2+4)} = \frac{1}{x-3} - \frac{x+3}{x^2+4}$

The process behind the method of Example 8 is as follows:

$B(x)$ is a product of distinct linear factors and a simple quadratic factor, such that:

$\frac{R(x)}{B(x)} = \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \ldots + \frac{c_n}{x-a_n} + \frac{dx+e}{x^2+bx+c}$, with degree $R(x) < n+2$. As for partial fractions with linear factors, here the numbers $c_1, c_2, \ldots c_n, d, e$ can be found by comparing coefficients or by combining that method with others.

In other words, $c_1, c_2, \ldots c_n$ can be found using any of the previous methods for partial fractions with linear factors. You then only need to find $d$ and $e$ as follows.

- If none of $a_1, a_2, \ldots a_n$ is zero, then let $x = 0$ to find the value of $e$.
- Multiply through by $x$ and let $x \to \infty$ (which will require dividing each fraction by the largest power of $x$ in the numerators) to find the value of $d$.
- If e.g. $a_1 = 0$, first find $d$ and then select a small integer value for $x$ distinct from $a_1, a_2, \ldots a_n$ to give a simple equation that can be solved for $e$.

## Example 9

Express $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)}$ using partial fractions.

### Solution

$x^2+x+1$ has no real linear factors; degree of numerator < degree of denominator.

Let $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)} = \dfrac{c}{x-2} + \dfrac{dx+e}{x^2+x+1}$

Write with common denominator: $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)} = \dfrac{c(x^2+x+1)+(dx+e)(x-2)}{(x-2)(x^2+x+1)}$

Write the numerators: $x^2+6x+5 \equiv c(x^2+x+1)+(dx+e)(x-2)$ [1]

Let $x=2$ in [1]: $21=7c$ $\quad \therefore c=3$

Let $x=0$ in [1]: $5=3-2e$ $\quad \therefore e=-1$

Equate coefficients of $x^2$ in [1]: $1=c+d$

$\therefore$ $1=3+d$ $\quad \therefore d=-2$

Hence: $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)} = \dfrac{3}{x-2} - \dfrac{2x+1}{x^2+x+1}$

## Example 10

Reduce $\dfrac{2x+7}{x(x^2+3x+1)}$ to its partial fractions.

### Solution

$x^2+3x+1$ has no real linear factors.

Let $\dfrac{2x+7}{x(x^2+3x+1)} = \dfrac{c}{x} + \dfrac{dx+e}{x^2+3x+1}$ [1]

$R(x)=2x+7 \qquad B(x)=x^3+3x^2+x \qquad B'(x)=3x^2+6x+1$

For $c$, let $x=0$: $c=\dfrac{R(0)}{B'(0)}=\dfrac{7}{1}=7$

Write [1] with common denominator: $\dfrac{2x+7}{x(x^2+3x+1)} = \dfrac{7(x^2+3x+1)+x(dx+e)}{x(x^2+3x+1)}$

Write the numerators: $2x+7 \equiv 7(x^2+3x+1)+x(dx+e)$

Let $x=1$: $9=35+d+e$ [2]

Let $x=-1$: $5=-7+d-e$ [3]

[2] + [3]: $14=28+2d$ $\quad \therefore d=-7$

Substitute into [2]: $-26=-7+e$ $\quad \therefore e=-19$

Hence: $\dfrac{2x+7}{x(x^2+3x+1)} = \dfrac{7}{x} - \dfrac{7x+19}{x^2+3x+1}$

## Example 11

Reduce $\frac{7}{(x^2+9)(x^2+16)}$ to its real partial fractions.

### Solution

Neither $x^2+9$ nor $x^2+16$ have real linear factors.

Let $\frac{7}{(x^2+9)(x^2+16)} = \frac{cx+d}{x^2+9} + \frac{ex+f}{x^2+16}$

| | | |
|---|---|---|
| Equate numerators: | $7 \equiv (cx+d)(x^2+16) + (ex+f)(x^2+9)$ | [1] |
| Let $x=0$ in [1]: | $7 = 16d + 9f$ | [2] |
| Let $x=1$ in [1]: | $7 = 17c + 17d + 10e + 10f$ | [3] |
| Let $x=-1$ in [1]: | $7 = -17c + 17d - 10e + 10f$ | [4] |
| Let $x=2$ in [1]: | $7 = 40c + 20d + 26e + 13f$ | [5] |
| [3] + [4]: | $14 = 34d + 20f$ | |
| | $7 = 17d + 10f$ | [6] |
| [6] − [2]: | $0 = d + f \quad \therefore f = -d$ | [7] |
| Substitute [7] into [2]: | $7 = 16d - 9d \quad \therefore d = 1 \quad \therefore f = -1$ | |
| [3] − [4]: | $0 = 34c + 20e \quad \therefore e = -1.7c$ | [8] |
| Substitute [8] into [5]: | $7 = 40c + 20 - 44.2c - 13 \quad \therefore c = 0 \quad \therefore e = 0$ | |

Hence: $\frac{7}{(x^2+9)(x^2+16)} = \frac{1}{x^2+9} - \frac{1}{x^2+16}$

With experience you will notice that $(x^2+16) - (x^2+9) = 7$, which is the numerator of the given fraction. This can make finding the partial fraction numerators easier with quadratic denominators of this type and a numerical numerator.

For fractions with quadratic denominators as in the example above, you can write:

$\frac{k}{(x^2+l)(x^2+m)} = \frac{a}{x^2+l} + \frac{b}{x^2+m}$ where $a$, $b$, $k$, $l$ and $m$ are constants.

This can be done because the original numerator ($k$) has no terms in $x$.

Equating numerators: $k = (a+b)x^2 + (am+bl)$

Hence $a = -b$ and $am + bl = k$. This means that $a$ and $b$ must have the same magnitude but opposite sign, so that:

$\frac{k}{(x^2+l)(x^2+m)} = \frac{a}{x^2+l} - \frac{a}{x^2+m}$ where $a = \frac{k}{m-l}$

**Example 12**

Reduce $\frac{x^2-6}{(x^2+9)(x^2+4)}$ to its real partial fractions.

**Solution**

Neither $x^2+9$ nor $x^2+4$ have real linear factors.

Because the numerator does not contain an odd power of $x$, it seems reasonable to assume that the numerator of each partial fraction will not contain a term with an odd power of $x$ (as was seen in the previous example). Try to use only $c$ and $d$ as the numerators of the partial fractions.

Let $\frac{x^2-6}{(x^2+9)(x^2+4)} = \frac{c}{x^2+9} + \frac{d}{x^2+4}$

$\therefore \quad \frac{x^2-6}{(x^2+9)(x^2+4)} = \frac{c(x^2+4)+d(x^2+9)}{(x^2+9)(x^2+4)}$

Equate numerators: $x^2-6 \equiv (c+d)x^2+4c+9d$

$\therefore \quad 1 = c+d \quad [1]$

$-6 = 4c+9d \quad [2]$

Rewrite [1]: $c = 1-d \quad [3]$

Substitute into [2]: $-6 = 4-4d+9d$

$5d = -10$

$d = -2$

Substitute into [3]: $c = 3$

Hence: $\frac{x^2-6}{(x^2+9)(x^2+4)} = \frac{3}{x^2+9} - \frac{2}{x^2+4}$

When finding partial fractions for pairs of quadratic denominators like $(x^2+a^2)$ and $(x^2+b^2)$ and the numerator does not have odd powers of $x$, Examples **11** and **12** show that you can write the first step as:

$\frac{cx^2+d}{(x^2+a^2)(x^2+b^2)} = \frac{e}{x^2+a^2} + \frac{f}{x^2+b^2}$ where either $c$ or $d$ could be zero.

It is an interesting task to show this algebraically.

### Repeated linear factors

This Mathematics Extension 2 course does not require you to find partial fractions with repeated linear factors, as in a question such as 'Reduce $\frac{1}{(x-1)(x-2)^2}$ to its linear factors'. However, you may be asked to show a repeated factors result such as in the next example.

**Example 13**

Show that: $\frac{1}{(x-1)(x-2)^2} = \frac{1}{x-1} - \frac{1}{x-2} + \frac{1}{(x-2)^2}$

**Solution**

$$\text{RHS} = \frac{1}{x-1} - \frac{1}{x-2} + \frac{1}{(x-2)^2}$$

$$= \frac{(x-2)^2-(x-1)(x-2)+(x-1)}{(x-1)(x-2)^2}$$

$$= \frac{x^2-4x+4-(x^2-3x+2)+x-1}{(x-1)(x-2)^2}$$

$$= \frac{1}{(x-1)(x-2)^2} = \text{LHS}$$