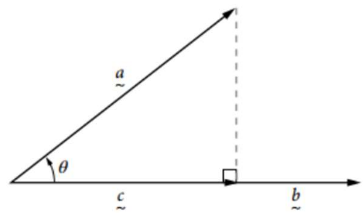
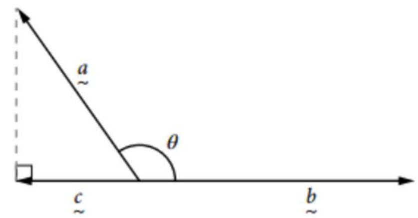
PROJECTIONS OF VECTORS

Any vector can be resolved into a sum of two vectors that are perpendicular to each other. It is usual to resolve the vector into two perpendicular components with one in a specified direction (e.g. parallel to a given vector) and the other perpendicular to that specified direction.

Scalar direction

In the diagrams, vector $\vec{c}$ is the vector projection of vector $\vec{a}$ in the direction of vector $\vec{b}$





The magnitude of $\vec{c}$, noted $|\vec{c}|$, is the scalar projection of vector $\vec{a}$ onto vector $\vec{b}$.

Using trigonometry, $\cos\theta = |\vec{c}| / |\vec{a}|$

By definition, $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ so $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\frac{|\vec{c}|}{|\vec{a}|} = |\vec{b}||\vec{c}|$

Therefore $|\vec{c}| = \frac{\vec{a}\,.\,\vec{b}}{|\vec{b}|}$

The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$, where $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{b}|}$

Example 21

Consider the vectors $\underset{\sim}{a} = 5\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$.

(a) Find the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(b) Find the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

Solution

(a) Find $\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$ by first finding $|\underset{\sim}{b}|$.

$$\underset{\sim}{b} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$$

$$|\underset{\sim}{b}| = \sqrt{3^2 + 4^2} = 5$$

$$\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{1}{5}(3\underset{\sim}{i} + 4\underset{\sim}{j})$$

Scalar projection $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$:

$$\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = (5\underset{\sim}{i} - \underset{\sim}{j}) \bullet \frac{1}{5}(3\underset{\sim}{i} + 4\underset{\sim}{j}) = \frac{15-4}{5} = \frac{11}{5}$$

The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\frac{11}{5}$.

(b) Find $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ by first finding $|\underset{\sim}{a}|$.

$$\underset{\sim}{a} = 5\underset{\sim}{i} - \underset{\sim}{j} : |\underset{\sim}{a}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$$

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{1}{\sqrt{26}}(5\underset{\sim}{i} - \underset{\sim}{j})$$

Scalar projection $\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}}$:

$$\underset{\sim}{b} \bullet \hat{\underset{\sim}{a}} = (3\underset{\sim}{i} + 4\underset{\sim}{j}) \bullet \frac{1}{\sqrt{26}}(5\underset{\sim}{i} - \underset{\sim}{j}) = \frac{15}{\sqrt{26}} - \frac{4}{\sqrt{26}} = \frac{11}{\sqrt{26}} = \frac{11\sqrt{26}}{26}$$

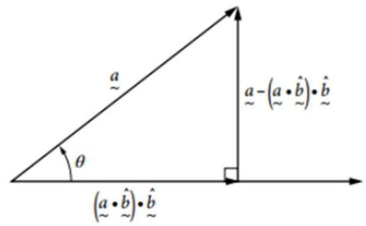
The scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is $\frac{11\sqrt{26}}{26}$

PROJECTIONS OF VECTORS

In general, the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ does *not* equal the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

Vector projection

In each of the diagrams above, vector $\underset{\sim}{c}$ is the **vector projection** of vector $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is a fraction of $\underset{\sim}{b}$, for example, $\frac{m}{n}\underset{\sim}{b}$ where m and n are real numbers.



Now, $\underset{\sim}{c} = |\underset{\sim}{c}|\hat{\underset{\sim}{b}}$
$= (\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$

The **vector projection** of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is: $\underset{\sim}{a}-\underset{\sim}{c}$
$= \underset{\sim}{a}-(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$

The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.
The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ can also be expressed as $\frac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\underset{\sim}{b}\bullet\underset{\sim}{b}}\underset{\sim}{b}$.
The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a}-(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$
The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ can also be expressed as $\underset{\sim}{a}-\frac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\underset{\sim}{b}\bullet\underset{\sim}{b}}\underset{\sim}{b}$.

Example 22

Consider the vectors $\underset{\sim}{a} = 2\underset{\sim}{i}-5\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i}+3\underset{\sim}{j}$.

(a) Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

(b) Find the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

Solution

(a) Find $\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$ by first finding $|\underset{\sim}{b}|$.

$\underset{\sim}{b} = -2\underset{\sim}{i}+3\underset{\sim}{j}: |\underset{\sim}{b}| = \sqrt{(-2)^2+3^2}$
$= \sqrt{13}$

$\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{1}{\sqrt{13}}(-2\underset{\sim}{i}+3\underset{\sim}{j})$

Scalar projection: $\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}}$:

$\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}} = (2\underset{\sim}{i}-5\underset{\sim}{j})\bullet\frac{1}{\sqrt{13}}(-2\underset{\sim}{i}+3\underset{\sim}{j})$
$= \frac{-4}{\sqrt{13}}+\frac{-15}{\sqrt{13}}$
$= \frac{-19}{\sqrt{13}}$

Vector projection: $(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$:

$(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{-19}{\sqrt{13}}\times\frac{1}{\sqrt{13}}(-2\underset{\sim}{i}+3\underset{\sim}{j})$
$= \frac{-19}{13}(-2\underset{\sim}{i}+3\underset{\sim}{j})$

The vector projection of $\underset{\sim}{a} = 2\underset{\sim}{i}-5\underset{\sim}{j}$ onto $\underset{\sim}{b} = -2\underset{\sim}{i}+3\underset{\sim}{j}$ is $\frac{-19}{13}(-2\underset{\sim}{i}+3\underset{\sim}{j})$.

(b) Find $\underset{\sim}{a}-(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$:

$\underset{\sim}{a}-(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = 2\underset{\sim}{i}-5\underset{\sim}{j}-\frac{-19}{13}(-2\underset{\sim}{i}+3\underset{\sim}{j})$
$= 2\underset{\sim}{i}-5\underset{\sim}{j}-\frac{38}{13}\underset{\sim}{i}+\frac{57}{13}\underset{\sim}{j}$
$= -\frac{12}{13}\underset{\sim}{i}-\frac{8}{13}\underset{\sim}{j}$
$= -\frac{4}{13}(3\underset{\sim}{i}+2\underset{\sim}{j})$

The vector projection of $\underset{\sim}{a} = 2\underset{\sim}{i}-5\underset{\sim}{j}$ perpendicular to $\underset{\sim}{b} = -2\underset{\sim}{i}+3\underset{\sim}{j}$ is $-\frac{4}{13}(3\underset{\sim}{i}+2\underset{\sim}{j})$.