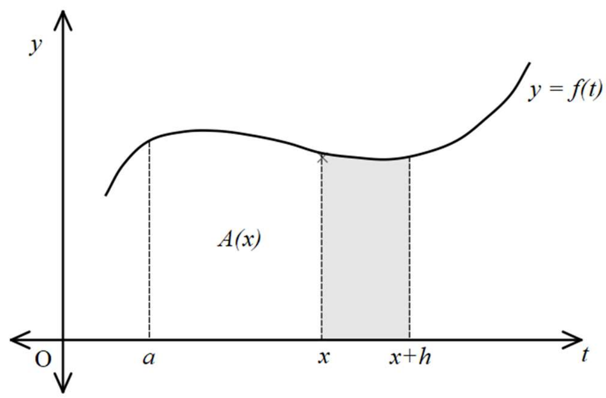
THE DEFINITE INTEGRAL AND THE PRIMITIVE FUNCTION

Fundamental theorem of calculus

Let $y = f(t)$ be a continuous positive curve defined for values of t

Let $A(x)$ be the area under the curve between a and x, and $A(x+h)$ be the area under the curve between a and $(x+h)$



The shaded area can be approximated by a rectangle of height $f(t)$ and base h

$$A(x+h) - A(x) \approx f(t) \times h$$

Therefore:

$$f(t) \approx \frac{A(x+h) - A(x)}{h}$$

For $x \le t \le x + h$, the minimum and maximum values of $f(t)$ are respectively m and M, i.e.:

$$m \le f(t) \le M \qquad \text{or} \qquad m \le \frac{A(x+h) - A(x)}{h} \le M$$

As $h \to 0$, both m and M tend towards $f(x)$, therefore: $\lim\limits_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$

But the left-hand side is also the derivative function of $A(x)$, therefore $A'(x) = f(x)$

So $A(x)$ is a primitive function of $f(x)$, i.e. $A(x) = F(x) + C$

When $x = a$ then $A(a) = 0$ so $F(a) + C = 0$ i.e. $C = -F(a)$ therefore: $A(x) = F(x) - F(a)$

$$\textit{Area under the curve } f(t) \textit{ between } a \textit{ and } x = F(x) - F(a)$$

This is known as the **fundamental theorem of calculus**

In fact, mathematicians had previously $\int_a^x f(t)\, dt$ to represent the area underneath the curve $f(t)$ between x and a, with the symbol $\int$ meaning the "Sum of all little rectangles of width dt and of height $f(t)$, for t varying between a and x", therefore:

$$\int_a^b f(t)\, dt = F(b) - F(a)$$

$F(b) - F(a)$ is sometimes also noted $[F(t)]_a^b$

Example 5

Evaluate:

(a) $\int_1^3 (x^2 - x)\,dx$ (b) $\int_{-1}^2 (x^3 - 2x^2 + 3x - 4)\,dx$

Solution

(a) $\int_1^3 (x^2 - x)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3$

$= \left(9 - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$

$= 4\frac{2}{3}$

(b) $\int_{-1}^2 (x^3 - 2x^2 + 3x - 4)\,dx = \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{3x^2}{2} - 4x\right]_{-1}^2$

$= \left(4 - \frac{16}{3} + 6 - 8\right) - \left(\frac{1}{4} + \frac{2}{3} + \frac{3}{2} + 4\right)$

$= -9\frac{3}{4}$

If you can find the primitive function you can now evaluate any definite integral. Unfortunately not all primitive functions are easy to find.

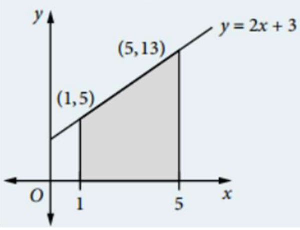
Definite integral of functions that cannot be integrated can be approximated using various methods (one of these methods is called the trapezoidal rule, which we'll use later).

Example 6

Calculate the area of the region bounded by the graph of the straight line $y = 2x + 3$, the x-axis and the ordinates $x = 1$ and $x = 5$.

Solution

Sketch the region:



Area $= \int_1^5 (2x + 3)\,dx$

$= \left[x^2 + 3x\right]_1^5$

$= (25 + 15) - (1 + 3)$

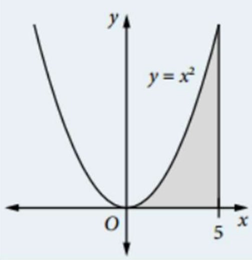
$= 36 \text{ units}^2$

Example 7

Use integration to find the area of the region enclosed by the parabola $y = x^2$, the x-axis and the line $x = 5$.

Solution

Sketch the region:



Area $= \int_0^5 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_0^5$

$= \frac{5^3}{3} - 0$

$= 41\frac{2}{3} \text{ units}^2$

Example 8

Calculate the area of the region bounded by:

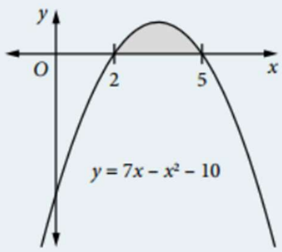
(a) the graph of the parabola $y = 7x - x^2 - 10$ and the x-axis

(b) the graph of the parabola $y = x^2 - 7x + 10$ and the x-axis.

(c) Compare the answers.

Solution

(a) Sketch the region:

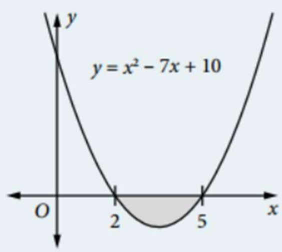


Cuts x-axis when $7x - x^2 - 10 = 0$:

$$x^2 - 7x + 10 = 0$$
$$(x-2)(x-5) = 0$$
$$x = 2, 5$$

$$\text{Area} = \int_2^5 \left(-x^2 + 7x - 10\right) dx$$
$$= \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5$$
$$= \left(-\frac{125}{3} + \frac{175}{2} - 50\right) - \left(-\frac{8}{3} + 14 - 20\right)$$
$$= 4.5$$

(b) Sketch the region:



Cuts x-axis when $x^2 - 7x + 10 = 0$:

$$(x-2)(x-5) = 0$$
$$x = 2, 5$$

$$\text{Area} = \int_2^5 \left(x^2 - 7x + 10\right) dx$$
$$= \left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x\right]_2^5$$
$$= \left(\frac{125}{3} - \frac{175}{2} + 50\right) - \left(\frac{8}{3} - 14 + 20\right)$$
$$= -4.5$$

Negative area does not make sense, but you can overcome this problem by taking the absolute value of the answer. The working needs to be written as:

$$\text{Area} = \left|\int_2^5 \left(x^2 - 7x + 10\right) dx\right| = |-4.5| = 4.5$$

(c) The two integrals have the same size but a different sign. The second curve and region are just the first curve and region reflected in the x-axis, so the regions must have the same area. The negative sign indicates that the region is below the x-axis.

Evaluating a definite integral and finding an area

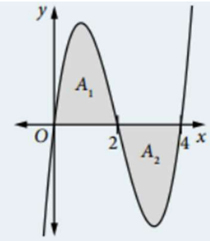
The value of an integral and the area of a region can have different signs, as Example 8 shows. When a region is entirely above the x-axis, the value of the integral is positive and is equal to the area. When the region is entirely below the x-axis, the value of the integral is negative. To find the area, you take the absolute value:

$$\text{Area} = \left|\int_a^b f(x)\,dx\right| = \left|\left[F(x)\right]_a^b\right| = |F(b) - F(a)|$$

What happens when the curve defining a region is both above and below the x-axis? In a situation like that you must find where the curve cuts the x-axis and then consider the regions above and below the x-axis separately.

Example 9

The curve $y = x^3 - 6x^2 + 8x$ cuts the x-axis at 0, 2, and 4 as shown in the diagram. By first finding the areas A_1 and A_2, find the total area enclosed by the curve and the x-axis.



Solution

$$\begin{aligned} A_1 &= \left|\int_0^2 \left(x^3 - 6x^2 + 8x\right)dx\right| \\ &= \left|\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2\right| \\ &= \left|(4 - 16 + 16) - (0)\right| \\ &= 4 \end{aligned}$$

$$\begin{aligned} A_2 &= \left|\int_2^4 \left(x^3 - 6x^2 + 8x\right)dx\right| \\ &= \left|\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4\right| \\ &= \left|(64 - 128 + 64) - (4 - 16 + 16)\right| \\ &= \left|-4\right| \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{Total area} &= A_1 + A_2 \\ &= 8 \text{ units}^2 \end{aligned}$$

Example 10

Find the value of the definite integral $\int_0^4 \left(x^3 - 6x^2 + 8x\right)dx$.

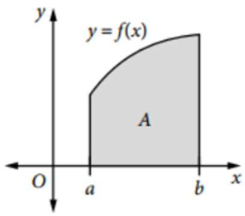
Solution

$$\begin{aligned} \int_0^4 \left(x^3 - 6x^2 + 8x\right)dx &= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^4 \\ &= (64 - 128 + 64) - (0) \\ &= 0 \end{aligned}$$

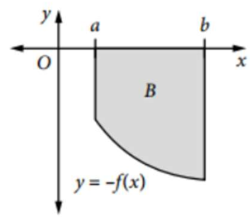
If you used this integral to find the area of the region in Example **9**, you would obtain an area of zero, which is clearly wrong. You should be aware that finding the value of a definite integral is not always the same as finding the area of a region. This is why it is important for you to sketch a region before setting up an integral to calculate its area. You must calculate the areas above and below the x-axis separately. Areas above and below the x-axis have opposite signs.

Important results

1 If $y = f(x)$ is an increasing function over the interval $a \le x \le b$, then $y = -f(x)$ is a decreasing function over the same interval. $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis. Thus the area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis is equal to the area of the region bounded by $y = -f(x)$, $x = a$, $x = b$ and the x-axis.



$$A = \left| \int_a^b f(x)\,dx \right|$$



$$B = \left| \int_a^b (-f(x))\,dx \right|$$

$$B = \left| -\int_a^b f(x)\,dx \right| = \left| \int_a^b f(x)\,dx \right|$$

2 If $y = f(x)$ is a continuous function over the interval $a \le x \le b$, and $a \le c \le b$, then:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

3 If F, G are primitives of f, g respectively, then $F + G$ is a primitive of $f + g$. Thus:

$$\begin{aligned} \int_a^b \left(f(x) + g(x)\right)dx &= \left(F(b) + G(b)\right) - \left(F(a) + G(a)\right) \\ &= \left(F(b) - F(a)\right) + \left(G(b) - G(a)\right) \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \end{aligned}$$

4 If you reverse the limits of integration, you change the sign of the integral: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5 $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b \left(f(x) \pm g(x)\right)dx$

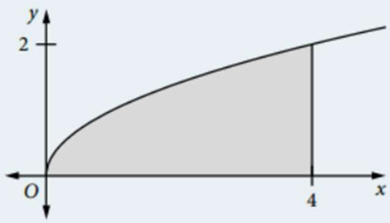
6 $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is a constant.

Example 11

Calculate the area between $y = \sqrt{x}$, the x-axis and $x = 4$.

Solution

Sketch the region:



It is clear from the diagram that the region is above the x-axis.

Hence: $$\text{Area} = \int_0^4 \sqrt{x}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{2}{3}(8-0)$$

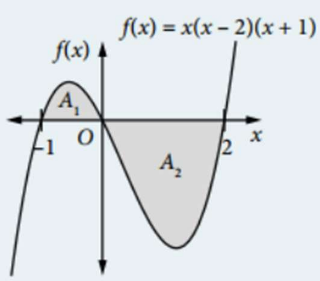
$$= 5\frac{1}{3}\text{ units}^2$$

Example 12

Find the area of the region bounded by the curve $f(x) = x(x-2)(x+1)$ and the x-axis.

Solution

Sketch the region:



$$f(x) = x(x-2)(x+1)$$
$$= x^3 - x^2 - 2x$$

$$A_1 = \int_{-1}^{0}\left(x^3 - x^2 - 2x\right)dx$$
$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}$$
$$= 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)$$
$$= \frac{5}{12}$$

$$A_2 = \left|\int_{0}^{2}\left(x^3 - x^2 - 2x\right)dx\right|$$
$$= \left|\left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}\right|$$
$$= \left|\left(\frac{16}{4} - \frac{8}{3} - 4\right) - 0\right|$$
$$= 2\frac{2}{3}$$

$$\text{Total area} = \frac{5}{12} + 2\frac{2}{3} = 3\frac{1}{12}\text{ units}^2$$

You can show by evaluating the integral that $\int_{-1}^{2}\left(x^3 - x^2 - 2x\right)dx \neq 3\frac{1}{12}$.