THE LANGUAGE AND LOGIC OF PROOF

1 Determine the negation of each of the following statements.

(a) p and q are both even.
(b) $x > 5$ or $x < -5$.
(c) x is divisible by either 7 or 8.
(d) $x = 0$ or $y = 0$.

a) p is odd OR q is odd

b) $x \leqslant 5$ AND $x \geqslant -5$, which can be simplified as $-5 \leqslant x \leqslant 5$

c) x is not divisible by 7 and x is not divisible by 8.
i.e. x is not divisible by 7 nor by 8.

d) $x \neq 0$ and $y \neq 0$
i.e. Neither x nor y are zero.

2 Translate the following statements into everyday language. Also, determine whether the statement is true or false, justifying your answer where appropriate.

(a) $\forall$ integers n, the number $2n + 3$ is odd.
(b) $\exists$ a real number x such that $\frac{1}{x} = x$.
(c) $\forall$ real numbers x, $x^2 > 0$.
(d) $\exists\, x \in R$ such that $x^2 = -1$.
(e) $\forall\, n \in$ integers, the number $n(n + 1)$ is divisible by 3.
(f) $\forall$ real numbers x and y, $x - y > 0$.
(g) $\forall$ real numbers x, $\exists$ a real number y such that $x + y = 0$.
(h) $\exists$ a real number x such that $\forall$ real numbers y, $xy = y$.

a) For all integer n, the number $(2n+3)$ is odd. TRUE

b) There exists a real number x such that $1/x = x$ TRUE

c) For all real numbers x, $x^2 > 0$ TRUE

d) There exists a real number x such that $x^2 = -1$ FALSE

e) For all n (an integer), the number $n(n+1)$ is divisible by 3
FALSE as it's not the case for $n = 4$

f) For all real numbers x and y, $x - y > 0$ FALSE

g) For all real number x, there exists a real number y such that $x + y = 0$. TRUE as $y = -x$

h) There exists a real number x such that, for any real number y, $xy = y$
TRUE as the possible number x is $x = 1$

3 Rewrite the following statements using the symbols ∀ and ∃. Also, determine whether the statement is true or false, justifying your answer where appropriate.

(a) The square of any integer is greater than the integer.
(b) There is a real number which, when multiplied by 5 gives an answer of 0.
(c) The sum of any two consecutive integers is odd.
(d) There is a real number equal to its square.
(e) The sum of the squares of any two real numbers is less than the product of the numbers.
(f) There is a special real number with the property that whenever another real number is divided by this special number, this other real number is obtained as a result.
(g) Every integer is divisible by at least one integer.

a) $\forall n \in \mathbb{Z}, \quad n^2 > n$ FALSE as for $n=1$ $1^2 > 1$ is false.

b) $\exists x \in \mathbb{R}$ such that $5x = 10$ TRUE $(x=2)$

c) $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}$ such that $n+(n+1) = 2k+1$ TRUE $(k=n)$

d) $\exists x \in \mathbb{R}$ such that $x^2 = x$ TRUE (for $x=0$ or $x=1$)

e) $\forall x \in \mathbb{R}$ and $\forall y \in \mathbb{R}, \quad x^2+y^2 < xy$

FALSE (example when $x>0$ and $y<0$)

f) $\exists x \in \mathbb{R}$ such that, $\forall y \in \mathbb{R}, \ \frac{y}{x} = y$

TRUE (as this is the case for $x=1$)

g) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ such that $\frac{n}{m} \in \mathbb{Z}$

TRUE as any integer is divisible by at least 2 numbers, namely 1 and itself.

4 Determine the negation of each of the following statements. Also state whether the original statement or the negation is true, justifying your answer where appropriate.

(a) $\forall$ real numbers x, $x^2 > 0$.
(b) $\exists$ a real number x such that $x^2 = x$.
(c) $\forall$ positive integers n, $10n > n$.
(d) $\forall$ real numbers x, x is either positive or negative.
(e) $\exists$ an integer n such that $n \neq 0$ and $n^2 < 1$.
(f) $\forall$ integers n, either $(-1)^n = 1$ or $(-1)^n = -1$.

a) $\exists x \in \mathbb{R}\ x^2 \leqslant 0$ (original statement is TRUE)

b) $\forall x \in \mathbb{R},\ x^2 \neq x$ (original statement is TRUE)

c) $\exists n \in \mathbb{Z}^+,\ 10n \leqslant n$ (original statement is TRUE)

d) $\exists x \in \mathbb{R}$, x neither positive nor negative)
(negation is TRUE as this is the case for $x = 0$)

e) $\forall n \in \mathbb{Z}$ either $n = 0$ or $n^2 \geqslant 1$ (negation is TRUE)

f) $\exists n \in \mathbb{Z}$ such that either $(-1)^n \neq 1$ and $(-1)^n \neq -1$
original statement is TRUE

5 Rewrite the following statements using the implication symbol $\Rightarrow$.

(a) If $x > 3$, then $x^2 > 9$.
(b) If n is divisible by 9, then n is divisible by 3.
(c) $n > 5$ implies that $n > 4$.
(d) $7p$ is positive if $p > 3$.
(e) q is even if $2q$ is a perfect square.
(f) m is a multiple of 6 is a sufficient condition to conclude that m is divisible by 3.
(g) It is necessary that $x^2 > 2$ if $x < -2$.
(h) n is even and greater than 2 is a sufficient condition to conclude that n is not prime.

a) $x > 3 \Rightarrow x^2 > 9$

b) n is divisible by 9 $\Rightarrow$ n is divisible by 3

c) $n > 5 \Rightarrow n > 4$ d) $p > 3 \Rightarrow 7p > 0$

e) $2q$ is a perfect square $\Rightarrow$ q is even.

f) m multiple of 6 $\Rightarrow$ m divisible by 3

g) $x < -2 \Rightarrow x^2 > 2$

h) n even and $n > 2$ $\Rightarrow$ n is not a prime

6 Write the converse, the contrapositive, and the negation of each of the following conditional statements. Determine whether each of the original, converse, contrapositive, and negation are true or false, justifying your answer where appropriate.

(a) If n is divisible by 20, then n is divisible by 5.
(b) If n is divisible by 3, then n^2 is divisible by 3.
(c) If $x > 7$, then $10x > 70$.
(d) If $xy = 0$, then either $x = 0$ or $y = 0$.

a) Original : TRUE
converse : if n divisible by 5, then n divisible by 20 FALSE
contrapositive : if n not divisible by 5, then n not divisible by 20 TRUE
Negation : $\exists\, n \in \mathbb{Z}$ such that n divisible by 20 and n not divisible by 5 . FALSE (as orignal was TRUE)

b) Original : TRUE
Converse : if n^2 divisible by 3, then n divisible by 3 TRUE
Contrapositive : if n^2 is not divisible by 3, then n is not divisible by 3 TRUE (as the original statement is true)
Negation : $\exists\, n \in \mathbb{Z}$ such that n divisible by 3 and n^2 not divisible by 3 FALSE (as original was true)

c) original = TRUE
Converse : if $10x > 70$, then $x > 7$ TRUE
Contrapositive : if $10x \leq 70$ then $x \leq 7$ TRUE
Negation : $\exists\, x \in \mathbb{R}$ such that $x > 7$ and $10x \leq 70$ FALSE

d) Original : TRUE
Converse : if $x = 0$ OR $y = 0$ then $xy = 0$ TRUE
Contrapositive : if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$ TRUE
Negation : $\exists\, x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $xy = 0$ and $x \neq 0$ and $y \neq 0$ FALSE (as original is TRUE)

7 Rewrite the following statements using the logical equivalence symbol, $\Leftrightarrow$.

(a) n is even if and only if n^2 is even.
(b) $x+y=0$ if and only if $x=-y$.
(c) n being even and divisible by 3 is necessary and sufficient for n to be divisible by 6.

a) n is even $\Leftrightarrow$ n^2 is even

b) $x+y=0 \Leftrightarrow x=-y$

c) n is even and divisible by 3 $\Leftrightarrow$ n divisible by 6

9

9 Which statement is true?

A $\forall x \in R, \exists y \in R$ such that $xy=6$.
B $\exists x \in R$ such that $\forall y \in R, xy=6$.
C $\exists x \in R$ such that $\forall y \in R, x+y=6$.
D $\forall x \in R, \exists y \in R$ such that $x+y=6$.

A not true for $x=0$ B not true for $y=0$
C Not true D TRUE.

10 The negation of the statement $\forall x \in R, \exists y \in R$ such that $x+y=6$, is:

A $\forall x \in R, \exists y \in R$ such that $x+y \neq 6$.
B $\exists x \in R$ such that $\forall y \in R, x+y \neq 6$.
C $\forall x \in R, \forall y \in R, x+y \neq 6$.
D $\exists x \in R, \exists y \in R$ such that $x+y \neq 6$.

11 Write the negation of the following statement, where x represents a real number: $x>0$ and $x<10 \Rightarrow x \geq 0$ and $x \leq 10$. Also, determine whether the original or the negation is true.

Original : TRUE

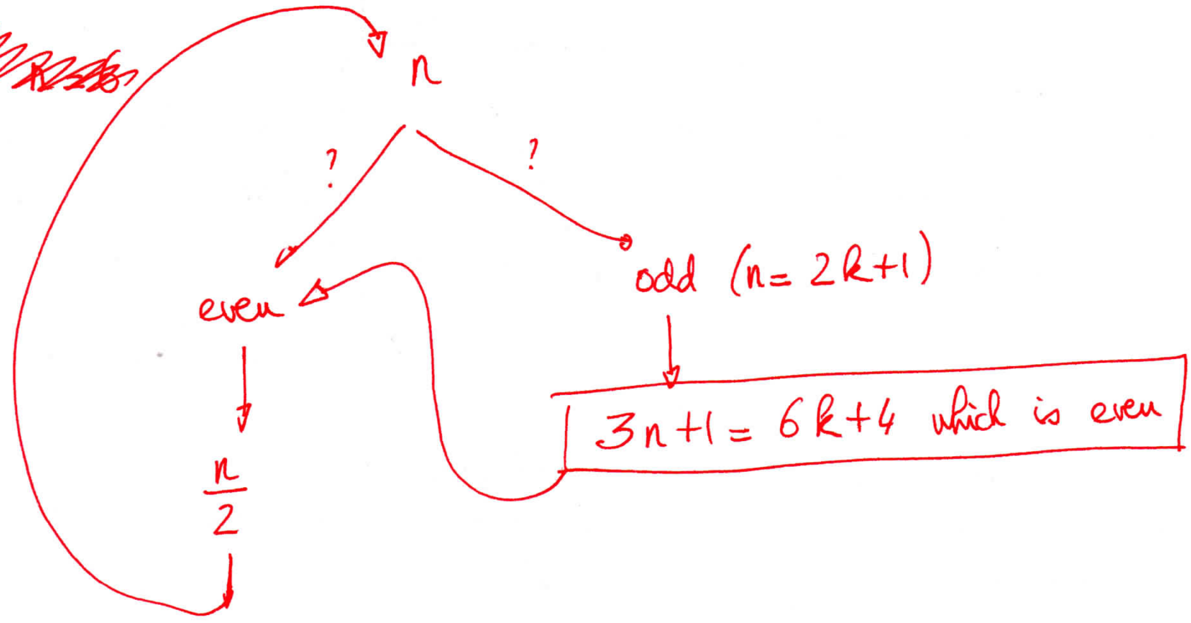
Negation: $\exists x \in \mathbb{R}$, such that $x>0$ and $x<10$ but $x<0$ or $x>10$ (Negation is FALSE)

12 Consider the following conjecture:
Start with any positive integer. If the integer is even, halve it. If the integer is odd, triple it and add one. Repeat this process. Eventually, the integer 1 will be obtained.
This is known as the '$3x + 1$' conjecture. It is yet to be proved but has been shown to be true for all integers up to roughly 10^{14}. Verify this conjecture for the following positive integers:

(a) 6 **(b)** 13 **(c)** 7

if $n = 2k$ (even) then $\frac{n}{2} = k$

if $n = 2k+1$ $3n+1 = 3(2k+1)+1 = 6k+4$ which is even.



a) $n = 6$ is even $\frac{n}{2} = 3$ which is odd

$3 \times 3 + 1 = 10$ which is even. $\frac{10}{2} = 5$ odd

$3 \times 5 + 1 = 16$ which is even $\frac{16}{2} = 8$ even

$\frac{8}{2} = 4$ even $\frac{4}{2} = 2$ even $\frac{2}{2} = 1$ ✓

b) $n = 13$ is odd $3 \times 13 + 1 = 40$

$\frac{40}{2} = 20$ $\frac{20}{2} = 10$ $\frac{10}{2} = 5$ back to a) so we'll end up with a 1 ✓

c) $n = 7$ is odd $3 \times 7 + 1 = 22$

$\frac{22}{2} = 11$ $3 \times 11 + 1 = 34$ $\frac{34}{2} = 17$ $3 \times 17 + 1 = 52$

$\frac{52}{2} = 26$ $\frac{26}{2} = 13$ back to b) so we'll end up with a 1 ✓