1 Use mathematical induction to prove **(c)** $\sum_{r=1}^{n}(r^2+1)r! = n\times(n+1)!$

Step 1 for $n=1$ $\sum_{r=1}^{1}(r^2+1)r! = (1^2+1)1! = 2$ whereas $1\times(1+1)! = 2$ so true

Step 2 We assume it's true for n, i.e $\sum_{r=1}^{n}(r^2+1)r! = n\times(n+1)!$

For $(n+1)$: $\sum_{r=1}^{n+1}(r^2+1)r! = \left[\sum_{r=1}^{n}(r^2+1)r!\right] + \left[(n+1)^2+1\right]\times(n+1)!$

$\_\_\_\_ = n\times(n+1)! + \left[(n+1)^2+1\right]\times(n+1)!$

$\_\_\_\_ = (n+1)!\left[n+(n^2+2n+2)\right]$

$\_\_\_\_ = (n+1)!\left[n^2+3n+2\right]$

The roots of the quadratic polynomial are (-1) and (-2), so

$\_\_\_\_ = (n+1)!\,(n+1)(n+2)$

$\_\_\_\_ = (n+1)\times(n+1)!\,(n+2) = (n+1)\times(n+2)!$

$\therefore$ it's true for $(n+1)$ if it's true for n.

Step 3. Conclusion: * it's true for $n=1$

* it's true for $(n+1)$ if it's true for n.

* $\therefore$ by induction, it's true for $n \geqslant 1$

2 **(a)** Simplify $\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}$. **(b)** Hence evaluate $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$.

(c) Use mathematical induction to prove that $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$ equals the result that you obtained in part **(b)**.

a) $\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}=\frac{1}{r+1}\times\left[\frac{1}{r}-\frac{1}{r+2}\right]$

$=\frac{1}{r+1}\left[\frac{(r+2)-r}{r(r+2)}\right]=\frac{2}{r(r+1)(r+2)}$

b) $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\left[\frac{1}{2}\left(\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right)\right]$

$=\frac{1}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right)\right]$

$=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{6}+\frac{1}{6}-\frac{1}{12}+\dots+\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right]$

$=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]=\frac{(n+1)(n+2)-2}{4(n+1)(n+2)}$

$=\frac{n^2+3n}{4(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

c) $n=1$ LHS $=\frac{1}{6}$ RHS $=\frac{1+3}{4\times 2\times 3}=\frac{4}{24}=\frac{1}{6}$ so true for $n=1$

Step 2: Assume it's true for n. $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

$P_{n+1}=\frac{n(n+3)}{4(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}=\frac{n(n+3)^2+4}{4(n+1)(n+2)(n+3)}$

$P_{n+1}=\frac{n^3+6n^2+9n+4}{4(n+1)(n+2)(n+3)}=\frac{(n+1)(n^2+5n+4)}{4(n+1)(n+2)(n+3)}=\frac{(n+4)(n+1)}{4(n+2)(n+3)}$

So $P_{n+1}=\frac{(n+1)\left[(n+1)+3\right]}{4(n+2)(n+3)}$ so true for $(n+1)$

Step 3: true for $n=1$
true for $(n+1)$ if true for n
$\therefore$ true $\forall\, n\geq 1$

3 Consider the sequence of numbers defined by $T_1 = 3$, $T_n = 2 \times T_{n-1} + 3$ for all $n \geq 2$.

(a) List the first five terms of this sequence.

(b) Prove by induction that $T_n = 3(2^n - 1)$ for all integers $n \geq 1$.

a) $T_1 = 3$ $\quad T_2 = 2 \times 3 + 3 = 9$ $\quad T_3 = 2 \times 6 + 3 = 15$

$T_4 = 2 \times 15 + 3 = 33$ $\quad T_5 = 2 \times 33 + 3 = 69$

b) Step 1 : $n = 1$ $\quad LHS = 3$ $\quad RHS = 3 \times (2^1 - 1) = 3$

so true for $n = 1$

Step 2 : Assume it's true for n. $T_n = 3(2^n - 1)$

$T_{n+1} = 2 \times T_n + 3 = 2 \times 3(2^n - 1) + 3$

$T_{n+1} = 3 \times 2(2^n - 1) + 3$

$T_{n+1} = 3[2(2^n - 1) + 1]$

$T_{n+1} = 3[2^{n+1} - 2 + 1]$

$T_{n+1} = 3[2^{n+1} - 1]$ so true for $(n+1)$ if true for n

Step 3 * it's true for $n = 1$

* it's true for $(n+1)$ if it's true for n.

* $\therefore$ by induction, it's true for all $n \geq 1$

4 (a) If $u_{n+1} = 2u_n + 1$ for all positive integral values of n, use mathematical induction to prove that $u_n + 1 = 2^{n-1}(u_1 + 1)$.

(b) If $u_1 = 1$, find the value of $\sum_{r=1}^{n} u_r$.

a) $u_{n+1} = 2u_n + 1$ for $n > 0$

Step 1 $n = 1$ LHS $= u_1 + 1$

RHS $= 2^{1-1}(u_1+1) = u_1 + 1$ so LHS = RHS true for $n = 1$

Step 2 We assume it's true for n.

In that case, $u_{n+1} + 1 = (2u_n + 1) + 1$

$= 2u_n + 2$

$= 2\left[2^{n-1}(u_1+1) - 1\right] + 2$

$= 2^n(u_1+1) - 2 + 2$

$= 2^n(u_1+1)$ so it's true for $(n+1)$ if it's true for n.

Step 3: Conclusion.

* it's true for $n=1$
* it's true for $(n+1)$ if it's true for n
* $\therefore$ by induction, it's true $\forall n \geq 1$

6 If $x > 0$ and $y > 0$, prove by induction that $(x+y)^n > x^n + y^n$ for all integers $n \geq 2$.

<u>Step 1</u> <u>$n=2$</u> LHS $= (x+y)^2 = x^2 + y^2 + 2xy$

RHS $= x^2 + y^2$ $\therefore$ as $2xy > 0$, then LHS > RHS

so it's true for $n=2$.

<u>Step 2</u>: we assume it's true for n, i.e. $(x+y)^n > x^n + y^n$

In that case $(x+y)^{n+1} = (x+y)^n (x+y)$

$\therefore (x+y)^{n+1} > (x+y)(x^n + y^n)$

$\therefore (x+y)^{n+1} > x^{n+1} + y^{n+1} + yx^n + xy^n$

As $yx^n > 0$ and $xy^n > 0$, then $yx^n + xy^n > 0$

$\therefore x^{n+1} + y^{n+1} + yx^n + xy^n > x^{n+1} + y^{n+1}$

$\therefore (x+y)^{n+1} > x^{n+1} + y^{n+1}$

So it's true for $(n+1)$ if it's true for n.

<u>Step 3</u>: it's true for $n=2$

it's true for $(n+1)$ if it's true for n

$\therefore$ by induction, it's true $\forall n \geq 2$

7 **(a)** By writing $\cos([2k+1]x)$ as $\cos(2kx+x)$, and remembering that $\cos 2x = 1 - 2\sin^2 x$, show that:

$$\frac{\sin 2kx}{2\sin x} + \cos(2k+1)x = \frac{\sin(2(k+1)x)}{2\sin x}$$

(b) Use the result of part **(a)** to prove by induction that $\cos x + \cos 3x + \ldots + \cos([2n-1]x) = \frac{\sin(2nx)}{2\sin x}$ for all positive integers n.

a) $$\frac{\sin 2kx}{2\sin x} + \cos(2k+1)x = \frac{\sin 2kx + 2\sin x \cos[(2k+1)x]}{2\sin x}$$

$$= \frac{\sin 2kx + 2\sin x \cos[2kx+x]}{2\sin x}$$

$$= \frac{\sin 2kx + 2\sin x[\cos(2kx)\cos x - \sin(2kx)\sin x]}{2\sin x}$$

$$= \frac{\sin 2kx[1-2\sin^2 x] + 2\sin x\cos x\cos(2kx)}{2\sin x}$$

$$= \frac{\sin(2kx)\times\cos(2x) + \sin(2x)\cos(2kx)}{2\sin x}$$

$$= \frac{\sin[2kx+2x]}{2\sin x} = \frac{\sin[2(k+1)x]}{2\sin x}$$

b) Step 1 : $n=1$ LHS $= \cos x$ RHS $= \frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$

So true for $n=1$

Step 2 : Assume it's true for n, then:

$$P_{n+1} = P_n + \cos[(2(n+1)-1)x] = \frac{\sin(2nx)}{2\sin x} + \cos[(2n+1)x]$$

$$P_{n+1} = \frac{\sin[2(n+1)x]}{2\sin x}$$ $\therefore$ true for $n+1$

Step 3 : * true for $n=1$ * true for $(n+1)$ if true for n

* $\therefore$ by induction, true $\forall\, n \geq 1$

8 If $x_1, x_2, x_3, \ldots x_n$ are positive real numbers, prove by induction that:
$(x_1 + x_2 + x_3 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}\right) \geq n^2$ for all integers $n \geq 1$.

Step 1 $n = 1$ $\quad$ LHS $= x_1 \times \frac{1}{x_1} = 1$ $\quad$ RHS $= 1^2$ so LHS $\geq$ RHS.

Step 2 We assume it's true for n.

i.e; $(x_1 + x_2 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right) \geq n^2$

For $(n+1)$ $\quad$ LHS $= \left[(x_1 + \ldots + x_n) + x_{n+1}\right]\left[\left(\frac{1}{x_1} + \ldots + \frac{1}{x_n}\right) + \frac{1}{x_{n+1}}\right]$

LHS $= (x_1 + x_2 + \ldots + x_n)\left(\frac{1}{x_1} + \ldots + \frac{1}{x_n}\right) + x_{n+1}\left(\frac{1}{x_1} + \ldots + \frac{1}{x_n}\right) + 1$
$\quad + \frac{1}{x_{n+1}}(x_1 + \ldots + x_n)$

$\therefore$ LHS $\geq n^2 + x_{n+1}\left(\frac{1}{x_1} + \ldots + \frac{1}{x_n}\right) + \frac{1}{x_{n+1}}(x_1 + \ldots + x_n) + 1$

$\therefore$ LHS $\geq n^2 + \left(\frac{x_{n+1}}{x_1} + \frac{x_{n+1}}{x_2} + \ldots + \frac{x_{n+1}}{x_n}\right) + \left(\frac{x_1}{x_{n+1}} + \frac{x_2}{x_{n+1}} + \ldots + \frac{x_n}{x_{n+1}}\right) + 1$

Now: , as $a + \frac{1}{a} \geq 2$, $\therefore$

LHS $\geq n^2 + 2 \times n + 1$

$\therefore$ LHS $\geq (n+1)^2$ $\quad$ So it's true for $(n+1)$ if it's true for n.

Step 3: * it's true for $n = 1$
* it's true for $(n+1)$ if it's true for n
* by induction, it's therefore true for all $n \geq 1$

9 **(a)** Prove that $x+\sqrt{x} \geq \sqrt{x(x+1)}$ for all real $x \geq 0$.

(b) A sequence is defined as $u_1 = 1$, $u_2 = 2$, $u_n = u_{n-1} + (n-1)u_{n-2}$ for $n \geq 3$. Prove by induction that $u_n \geq \sqrt{n!}$.

a) Both sides are positive as $x \geq 0$.

So $x+\sqrt{x} \geq \sqrt{x(x+1)} \iff (x+\sqrt{x})^2 \geq \left(\sqrt{x(x+1)}\right)^2$

$\iff x^2 + x + 2x\sqrt{x} \geq x(x+1)$

$\iff x^2 + x + 2x\sqrt{x} \geq x^2 + x$

$\iff 2x\sqrt{x} \geq 0$ which is true as $x \geq 0$

$\therefore x+\sqrt{x} \geq \sqrt{x(x+1)} \quad \forall x \geq 0$

b) $u_1 = 1$ LHS $= 1$ whereas RHS $= \sqrt{1!} = 1$

$\therefore u_n \geq \sqrt{n!}$ for $n=1$ also true for $n=2$ as $2 \geq \sqrt{2!}$

Step 2: We assume it's true for n.

For $(n+1)$: $u_{n+1} = u_n + n \times u_{n-1}$

$\therefore$, as $u_n \geq \sqrt{n!}$ and $u_{n-1} \geq \sqrt{(n-1)!}$

then $u_n + n\, u_{n-1} \geq \sqrt{n!} + n\sqrt{(n-1)!}$

So $u_{n+1} \geq \sqrt{n!} + n \times \sqrt{(n-1)!}$

$\iff u_{n+1} \geq \sqrt{(n-1)! \times n} + n\sqrt{(n-1)!}$

$\iff u_{n+1} \geq \sqrt{(n-1)!}\,[\sqrt{n} + n]$

$\iff u_{n+1} \geq \sqrt{(n-1)!}\,[\sqrt{n}\,(1+\sqrt{n})]$

But $1+\sqrt{n} > \sqrt{n+1} \quad \forall n \in \mathbb{N}$, therefore

$u_{n+1} \geq \sqrt{(n-1)!} \times \sqrt{n} \times \sqrt{n+1} \quad \therefore u_{n+1} \geq \sqrt{(n+1)!}$

$\therefore$ it's true for $n+1$

Step 3: * true for $n=1$ * true for $n+1$ if true for n

$\therefore$ by induction, it's true $\forall n \in \mathbb{N}$.