ARITHMETIC SERIES

1 Find the sum of the first 16 terms of the arithmetic series $3 + 4\frac{1}{4} + 5\frac{1}{2} + \ldots$

$4\frac{1}{4} - 3 = 1\frac{1}{4} = 5\frac{1}{2} - 4\frac{1}{4}$ So $T_n = 3 + (n-1)1\frac{1}{4}$

So $a = 3$ and $d = 1\frac{1}{4}$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$\therefore S_{16} = \frac{16}{2}\{2 \times 3 + (16-1) \times 1\tfrac{1}{4}\}$$

$$S_{16} = 198$$

4 The first three terms of an arithmetic series are $-2 + 3 + 8 + \ldots$

(a) Find the 60th term. **(b)** Hence, or otherwise, find the sum of the first 60 terms of the series.

$d = 5$ $a = -2$ so $T_n = -2 + (n-1) \times 5 = 5n - 7$

a) $T_{60} = 5 \times 60 - 7 = 293$

b) $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$$S_{60} = \frac{60}{2}\{2 \times (-2) + (60-1) \times 5\}$$

$$S_{60} = 8{,}730$$

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6 An object falling freely from a height travels 4.9 metres in the first second, 14.7 metres in the second second and 24.5 metres in the third second. How far has it fallen:

(a) after six seconds **(b)** between the fifth and the sixth second?

a) $d = 14.7 - 4.9 = 24.5 - 14.7 = 9.8$

So $T_n = 4.9 + (n-1) \times 9.8 = 9.8n - 4.9$

$S_6 = \frac{6}{2}\left[2 \times (4.9) + (6-1) \times 9.8\right]$

$\therefore S_6 = 176.4$ m

b) $T_6 = 9.8 \times 6 - 4.9 = 53.9$ m

8 Find the sum of the first 20 terms of an arithmetic series whose eighth term is 6 and whose twelfth term is 9.

$T_8 = 6 \qquad T_{12} = 9 \qquad \therefore d = \frac{T_{12} - T_8}{12 - 8} = \frac{9-6}{4} = \frac{3}{4}$

But $T_8 = T_1 + (8-1) \times \frac{3}{4} = 6$

$\therefore T_1 = 6 - 7 \times \frac{3}{4} = 0.75$

$\therefore S_n = \frac{n}{2}\left[2 \times 0.75 + (n-1) \times \frac{3}{4}\right]$

$\therefore S_{20} = \frac{20}{2}\left[2 \times 0.75 + (20-1) \times \frac{3}{4}\right]$

$S_{20} = 157.5$

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10 Evaluate: (a) $\sum_{k=1}^{10}(3k-7)$ (b) $\sum_{k=1}^{8}(4k+1)$ (c) $\sum_{k=1}^{n}(4k-1)$

a) $T_1 = -4 \quad d = +3 \quad \therefore S_n = \frac{n}{2}\left[2\times(-4)+(n-1)d\right]$

$S_{10} = \frac{10}{2}\left[-8+9\times(+3)\right] = 95$

b) $T_1 = 5 \quad d = 4 \quad \therefore S_8 = \frac{8}{2}\left[2\times 5+(8-1)4\right] = 152$

c) $T_1 = 3 \quad d = 4$

$\therefore S_n = \frac{n}{2}\left[2\times 3 + (n-1)4\right] = \frac{n}{2}\left[6+4n-4\right]$

$\therefore S_n = \frac{n}{2}\left[4n+2\right] = n\left[2n+1\right]$

11 The first term of an arithmetic series is 7, the common difference is 2 and the sum of the first n terms is 247. Find the value of n.

$T_1 = 7 \quad d = 2 \quad S_n = \frac{n}{2}\left[2\times 7+(n-1)2\right] = 247$

$\therefore \quad n(14+2n-2) = 494$

$\Leftrightarrow \quad n(2n+12) = 494$

$\Leftrightarrow \quad n(n+6) = 247 \Leftrightarrow n^2+6n-247=0$

$\Delta = 36 + 4\times(247) = 32^2$

$\therefore n = \frac{-6+32}{2} = 13$

14 Find the sum of the integers between 0 and 101 that are:

(a) divisible by 2 **(b)** divisible by 5 **(c)** divisible by 2 and 5 **(d)** divisible by 2 or 5 but not both.

a) This is $\sum_{n=0}^{50} 2n = 2\sum_{n=0}^{50} n = 2 \times \frac{50(50+1)}{2} = 2550$

b) This is $\sum_{n=0}^{20} 5n = 5\sum_{n=0}^{20} n = 5 \times \frac{20(20+1)}{2} = 1050$

c) This is $\sum_{n=0}^{10} 10n = 10\sum_{n=0}^{10} n = 10 \times \frac{10(10+1)}{2} = 550$

d) This sum is $2+4+5+6+8+12+14+15+16+18+22+24+25+26+28+\dots$

$S = [2+4+6+8+12+14+16+18+22+24+26+28+\dots] + [5+15+25+\dots]$

$S = [2550 - 550] + [1050 - 550]$

$S = 2{,}500$

16 The sum of the magnitudes of the angles of an irregular pentagon (five-sided polygon) is 540°. The magnitudes of the angles form the terms of an arithmetic series. If the largest angle has a magnitude of 136°, find the magnitude of each of the other four angles.

$S_5 = 540$ $\quad T_n = T_1 + (n-1)d$ $\quad T_5 = 136 = T_1 + (5-1)d$

$S_5 = \frac{5}{2}[2T_1 + (5-1)d] = \frac{5}{2}[2T_1 + 4d]$ $\quad T_5 = T_1 + 4d$

$\therefore 136 = T_1 + 4d$

$\therefore 4d = 136 - T_1$

$540 = \frac{5}{2}[2T_1 + (136 - T_1)]$

$\therefore 216 = [T_1 + 136]$

$\therefore T_1 = 80$ so $d = \frac{136 - T_1}{4} = \frac{136 - 80}{4} = 14$

$T_2 = 94 \quad T_3 = 108 \quad T_4 = 122$ and $T_5 = 136$

21 The track of a vinyl record is in the shape of a spiral curve, but it may be considered as a number of concentric circles of minimum and maximum radius 5.25 cm and 10.5 cm respectively. The record rotates at $33\frac{1}{3}$ revolutions per minute and takes 18 minutes to play from start to finish. Find an approximation to the length of the track.

So No of revolutions $= 18 \times 33\frac{1}{3} = 600$

$T_1 = 2\pi \times 5.25 \qquad T_{600} = 2\pi \times 10.5$

$d \simeq \dfrac{T_{600} - T_1}{600 - 1} = \dfrac{2\pi \times 10.5 - 2\pi \times 5.25}{599} = 2\pi \times \dfrac{21}{2396} = \dfrac{21\pi}{1198}$

$S_{600} = \dfrac{600}{2}\left[2 \times T_1 + (600-1) \times \dfrac{21\pi}{1198}\right]$

$S_{600} = 300\left[21\pi + 599 \times \dfrac{21\pi}{1198}\right]$

$S_{600} = 6300\pi\left[\dfrac{3}{2}\right] = 9450\pi \simeq 297\text{ m}$

22 Given $S_n = 3n^2 - 11n$, find T_n and hence show that the series is arithmetic.

$S_n = 3n^2 - 11n$

$S_{n-1} = 3(n-1)^2 - 11(n-1)$

$\therefore S_n - S_{n-1} = [3n^2 - 11n] - [3n^2 - 17n + 14]$

$\qquad = 6n - 14 = T_n$

$\therefore T_1 = 6 - 14 = -8$

$T_2 = 12 - 14 = -2$

Generally $T_n - T_{n-1} = (6n-14) - (6(n-1) - 14)$

$\qquad = 6n - 14 - (6n - 20)$

$\qquad = 6 = d$ $\therefore$ the series is arithmetic as d is a constant.

27 Find the first three terms of an arithmetic series in which the fifth term is three times the second term, and the sum of the first six terms is 36.

$T_5 = 3 \times T_2 \qquad S_6 = 36 = \frac{6}{2}\left[2T_1 + (6-1)d\right]$

$d = \frac{T_5 - T_2}{5-2} = \frac{3T_2 - T_2}{3} = \frac{2T_2}{3}$ also $T_2 = T_1 + d$.

so $T_1 = T_2 - d$

So $36 = 3\left[\left(T_2 - \frac{2T_2}{3}\right)2 + 5 \times \frac{2T_2}{3}\right]$

$12 = \frac{\cancel{4}2}{3}T_2 + \frac{10}{3}T_2 \qquad \therefore T_2 = \frac{36}{12} = 3$

$\therefore d = \frac{2}{3} \times 3 = 2$

$\therefore T_1 = 3 - 2 = 1 \qquad T_2 = 3$ and $T_3 = 5$

28 Logs of wood are stacked in a pile so that there are 15 logs on the top row, 16 on the next row, 17 on the next, and so on. If there are 246 logs altogether:

(a) how many rows are there **(b)** how many logs are on the bottom row?

$S_n = 246 = \frac{n}{2}\left[2 \times 15 + (n-1) \times 1\right] = \frac{n}{2}\left[n + 29\right]$

$\therefore n^2 + 29n = 492 \qquad n = \frac{-29 + 53}{2} = 12$ rows.

$T_n = T_1 + (n-1) \times 1 = 15 + n - 1 = n + 14$

So $T_{12} = 12 + 14 = 26$

30 How many terms of the series 6 + 10 + 14 + … must be taken to give a sum of 880?

$T_1 = 6 \quad d = 4$

$$S_n = \frac{n}{2}\left[2\times 6 + (n-1)\times 4\right] = \frac{n}{2}\left[4n+8\right] = n(2n+4)$$

$$2n^2 + 4n - 880 = 0 \iff n^2 + 2n - 440 = 0$$

$$\Delta = 4 + 4\times 440 = 42^2 \qquad n = \frac{-2+42}{2} = 20$$

So 20 terms.

32 Find the sum of: **(a)** the first n odd positive integers **(b)** the first n even positive integers **(c)** the first n positive integers, and find this value of n if the sum is 210.

a) This is $\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n = n^2$

b) This is $\sum_{k=1}^{n} 2k = 2\sum_{k=1}^{n} k = 2\times\frac{n(n+1)}{2} = n(n+1) = n^2+n$

c) This is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

is $\frac{n(n+1)}{2} = 210$ then $n^2 + n - 420 = 0$

$$\Delta = 1 + 4\times 420 = 41^2 \qquad n = \frac{-1+41}{2} = 20$$

so $n = 20$

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34 Cans of fruit in a supermarket display are stacked so that there are 3 cans in the top row, 5 in the next row, 7 in the next row and so on. If there are 10 rows in the display, find:

(a) the number of cans in the bottom row **(b)** the total number of cans in the display.

a) $T_n = T_1 + (n-1)d = 3 + (n-1)2 = 2n+1$

$T_{10} = 2 \times 10 + 1 = 21$

b) $S_n = \frac{n}{2}\left[2T_1 + (n-1)d\right]$

$S_{10} = \frac{10}{2}\left[2\times 3 + (10-1)\times 2\right] = 5\left[6 + 18\right]$

$S_{10} = 120$ cans.

35 The first term of an arithmetic series is 5. The ratio of the sum of the first four terms to the sum of the first ten terms is 8 : 35. Find the common difference.

$T_1 = 5$ $\quad \frac{S_4}{S_{10}} = \frac{8}{35} = \frac{\frac{4}{2}\left[2\times 5 + (4-1)d\right]}{\frac{10}{2}\left[2\times 5 + (10-1)d\right]} = \frac{2(10+3d)}{5(10+9d)}$

$\therefore\ 40(10+9d) = 35\times 2(10+3d)$

$\Leftrightarrow\ 4(10+9d) = 7(10+3d)$

$\Leftrightarrow\ 15d = 70 - 40 = 30 \qquad \therefore\ d = 2$