DERIVATIVES OF $f(x)=c$, $f(x)=x$, $f(x)=x^2$, $f(x)=x^n$
DERIVATIVE OF THE SUM OF TWO FUNCTIONS

1 Find the derivative of:

(a) $y = 3x^2 + 2x - 1$

$y' = 6x + 2$

(b) $y = 4x - 3x^2$

$y' = 4 - 6x$

(c) $y = 7x - 4x^2$

$y' = 7 - 8x$

(d) $y = x^4 + x^2 + 1$

$y' = 4x^3 + 2x$

(e) $y = x^5 - x^3 + x$

$y' = 5x^4 - 3x^2 + 1$

(f) $v = t^3 + 4t^2 - 2t + 5$

$v' = 3t^2 + 8t - 2$

2 Find the derivative of:

(a) $y = x^{\frac{3}{2}}$ **(b)** $y = \frac{2}{x}$ **(c)** $y = 2\sqrt{x}$ **(d)** $v = \sqrt[3]{t^2}$ **(e)** $h(m) = \frac{1}{m^3}$ **(f)** $f(x) = \frac{1}{\sqrt{x}}$

a) $y' = \frac{3}{2} x^{3/2 - 1} = \frac{3}{2} x^{1/2} = \frac{3}{2}\sqrt{x}$

b) $y = 2x^{-1}$ so $y' = 2 \times (-1) x^{-1-1} = -2x^{-2} = \frac{-2}{x^2}$

c) $y = 2x^{1/2}$ so $y' = 2 \times \frac{1}{2} \times x^{\frac{1}{2} - 1} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

d) $v = t^{2/3}$ so $v' = \frac{2}{3} t^{\frac{2}{3} - 1} = \frac{2}{3} t^{-\frac{1}{3}} = \frac{2}{3t^{\frac{1}{3}}} = \frac{2}{3\sqrt[3]{t}}$

e) $h(m) = m^{-3}$ so $h'(m) = -3m^{-3-1} = -3m^{-4} = \frac{-3}{m^4}$

f) $f(x) = \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$

so $f'(x) = -\frac{1}{2} x^{-\frac{1}{2} - 1} = -\frac{1}{2} x^{-3/2} = -\frac{1}{2x^{3/2}} = -\frac{1}{2x\sqrt{x}}$

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4 Expand each expression and find $\frac{dy}{dx}$.

(a) $y=(x-1)(x+2)$ **(b)** $y=3x(x^2-2)$ **(c)** $y=(2x-3)^2$
(d) $y=(x-4)(x+4)$ **(e)** $y=(2x-3)^3$ **(f)** $y=(x-2)(x+1)(3x+1)$

a) $y=x^2-x+2x-2=x^2+x-2$

so $\frac{dy}{dx}=2x+1$

b) $y=3x(x^2-2)=3x^3-6x$

so $\frac{dy}{dx}=9x^2-6$

c) $y=(2x-3)^2=4x^2-12x+9$ so $\frac{dy}{dx}=8x-12$

d) $y=(x-4)(x+4)=x^2-16$ so $\frac{dy}{dx}=2x$

e) $y=(2x-3)^3=8x^3-3\times(2x)^2\times3+3\times(2x)\times3^2-3^3$

$y=8x^3-36x^2+54x-27$ so $\frac{dy}{dx}=24x^2-72x+54$

f) $y=(x-2)(x+1)(3x+1)$

$y=(x-2)(3x^2+x+3x+1)=(x-2)(3x^2+4x+1)$

$y=3x^3+4x^2+x-6x^2-8x-2$

$y=3x^3-2x^2-7x-2$

so $\frac{dy}{dx}=9x^2-4x-7$

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6 Find $f'(x)$.

(a) $f(x)=x+\sqrt{x}$ **(b)** $f(x)=x^2+\frac{1}{x}$ **(c)** $f(x)=x^2+x+1+\frac{1}{x}+\frac{1}{x^2}$

(d) $f(x)=x^{\frac{2}{3}}+x^{\frac{1}{3}}$ **(e)** $f(x)=\left(x-\frac{1}{x}\right)^2$ **(f)** $f(x)=x\sqrt{x}$

a) $f'(x)=1+\frac{1}{2}x^{\frac{1}{2}-1}=1+\frac{1}{2}x^{-1/2}=1+\frac{1}{2\sqrt{x}}$

b) $f'(x)=2x-x^{-1-1}=2x-x^{-2}=2x-\frac{1}{x^2}$

c) $f(x)=x^2+x+1+x^{-1}+x^{-2}$

so $f'(x)=2x+1-1\times x^{-1-1}-2x^{-2-1}=2x+1-\frac{1}{x^2}-\frac{2}{x^3}$

d) $f'(x)=\frac{2}{3}x^{\frac{2}{3}-1}+\frac{1}{3}x^{\frac{1}{3}-1}=\frac{2}{3}x^{-1/3}+\frac{1}{3}x^{-2/3}=\frac{2}{3x^{1/3}}+\frac{1}{3x^{2/3}}$

e) $f(x)=x^2-2x\times\frac{1}{x}+\frac{1}{x^2}=x^2-2+x^{-2}$

so $f'(x)=2x-2x^{-2-1}=2x-\frac{2}{x^3}$

f) $f(x)=x\sqrt{x}=x\times x^{1/2}=x^{3/2}$

so $f'(x)=\frac{3}{2}x^{\frac{3}{2}-1}=\frac{3}{2}x^{\frac{1}{2}}=\frac{3}{2}\sqrt{x}$

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7 For $f(x) = 3x^2 - 2x + 7$, indicate whether each statement is correct or incorrect.

(a) $f'(x) = 6x - 2$ TRUE **(b)** $f'(0) = 7$ NO **(c)** $f(1) = 8$ TRUE **(d)** $f'(2) = 10$ TRUE

$f'(x) = 6x - 2$ so a/ correct

$f'(0) = 6 \times 0 - 2 = -2$ $\quad f(1) = 3 \times 1^2 - 2 \times 1 + 7 = 3 - 2 + 7 = 8$

$f'(2) = 6 \times 2 - 2 = 12 - 2 = 10$

8 For each of the following functions, find the value of x for which $f'(x) = 0$.

(a) $f(x) = x^2 - 4$ **(b)** $f(x) = 2x^3 - 6x$ **(c)** $f(x) = x^3 - 4x^2$

a) $f'(x) = 2x$ so $f'(x) = 0$ when $2x = 0$, i.e when $x = 0$

b) $f'(x) = 2 \times 3x^2 - 6 = 6x^2 - 6 = 6(x^2 - 1)$

so $f'(x) = 0$ when $x^2 - 1 = 0$, i.e when $x = \pm 1$

c) $f'(x) = 3x^2 - 4 \times 2x = 3x^2 - 8x = x(3x - 8)$

so $f'(x) = 0$ when $3x - 8 = 0$, i.e when $x = 8/3$

$x = 0$

2 values : $x = 0$ and $x = 8/3$ give $f'(x) = 0$

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9 Find the gradient of the curve $y = x^2 - x - 6$ at the points where $y = 0$.

$f(x) = x^2 - x - 6$ so $f'(x) = 2x - 1$

when $y = 0$, $x^2 - x - 6 = 0$ $\Delta = 1^2 - 4\times(-6) = 25$

so two roots $x_1 = \frac{1-\sqrt{25}}{2} = \frac{1}{2} - \frac{5}{2} = -2$ and $x_2 = \frac{1}{2} + \frac{5}{2} = 3$

$f'(2) = 2\times(-2) - 1 = -5$

$f'(3) = 2\times3 - 1 = 5$

11 Show that the graph of $y = x^2 + 4x - 12$ crosses the x-axis at two points. Find the gradient of the curve at these points.

$x^2 + 4x - 12 = 0$ $\Delta = 4^2 - 4\times(-12) = 64$ so two roots

$x_1 = \frac{-4-8}{2} = -6$ and $x_2 = \frac{-4+8}{2} = 2$

$f'(x) = 2x + 4$

$f'(-6) = 2\times(-6) + 4 = -12 + 4 = -8$

$f'(2) = 2\times(2) + 4 = 4 + 4 = 8$

13 Find the coordinates of the points on the curve $y=x^2-5x+6$ at which the tangent:

(a) makes an angle of 45° with the x-axis
(b) is parallel to the line with equation $3x+y-4=0$
(c) is perpendicular to the line with equation $2y-x+3=0$.

a) an angle of 45° corresponds to a gradient of 1

$\frac{dy}{dx}=2x-5$ so $\frac{dy}{dx}=1$ when $2x-5=1$
$\Leftrightarrow 2x=6$ so $x=3$

At $x=3$, the gradient is 1 $\quad f(3)=3^2-5\times3+6=0$
the coordinates of this point are $(3,0)$

b) The gradient of this line is -3

$\frac{dy}{dx}=-3$ when $2x-5=-3$ so $2x=2$ so $x=1$

$f(1)=1^2-5\times1+6=1-5+6=2$

At $(1,2)$ the gradient is -3, like for $3x+y-4=0$

c) The gradient of this line is $\frac{1}{2}$, so the normal has for gradient -2.

$\frac{dy}{dx}=-2$ when $2x-5=-2$, i.e. $2x=3$

so at $x=\frac{3}{2}$ $\quad f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^2-5\times\frac{3}{2}+6=\frac{3}{4}$

At $\left(\frac{3}{2},\frac{3}{4}\right)$, the tangent to the curve is perpendicular to the line of equation $2y-x+3=0$

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17 The profit function, in dollars, for a manufacturer is given by the function $P = 6x - \frac{x^2}{2} - 10$, where x is the number of items produced in a day up to a maximum of 6 items.

(a) If the break-even point is when the profit is zero, what is the break-even point for this manufacturer?

(b) Find $\frac{dP}{dx}$.

(c) For what values of x is $\frac{dP}{dx} > 0$?

a) $6x - \frac{x^2}{2} - 10 = 0 \iff 12x - x^2 - 20 = 0$

$\iff -x^2 + 12x - 20 = 0 \iff x^2 - 12x + 20 = 0$

$\Delta = 12^2 - 4 \times 20 = 64$ two roots

$x_1 = \frac{12+8}{2} = 10$ and $x_2 = \frac{12-8}{2} = \frac{4}{2} = 2$ (lowest)

The break even point is when 2 items are produced in a day.

b) $\frac{dP}{dx} = 6 - \frac{2x}{2} = 6 - x$

c) $\frac{dP}{dx} > 0$ when $6 - x > 0$, i.e. when $x < 6$.

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20 **(a)** Given $f(x) = x^2 + 3$, find $f'(x)$.

(b) On the same diagram sketch the graph of $y = f(x)$ and $y = f'(x)$.

a) $f'(x) = 2x$