ZEROS OF A POLYNOMIAL

Conjugate root theorem

If a polynomial $P(z)$ has real coefficients and if $P(a+ib)=0$ then $P(a-ib)=0$ i.e. complex zeros of polynomials with real coefficients occur in conjugate pairs.

Proof: Consider $P(z)=a_n\,z^n+a_{n-1}\,z^{n-1}+\cdots+a_1\,z+a_0$

$P(a+ib)=0$ means $a_n\,(a+ib)^n+a_{n-1}\,(a+ib)^{n-1}+\cdots+a_1\,(a+ib)+a_0=0$

Or if we use the polar form of the complex number $a+ib=r\,e^{i\theta}$

$$a_n\left(r\,e^{i\theta}\right)^n+a_{n-1}\left(r\,e^{i\theta}\right)^{n-1}+\cdots+a_1\left(r\,e^{i\theta}\right)+a_0=0$$

or

$$a_n\,r^n\,e^{in\theta}+a_{n-1}\,r^{n-1}\,e^{i(n-1)\theta}+\cdots+a_1\,r\,e^{i\theta}+a_0=0$$

We can take the conjugate of both sides.

$$\overline{a_n\,r^n\,e^{\imath n\theta}+a_{n-1}\,r^{n-1}\,e^{\imath(n-1)\theta}+\cdots+a_1\,r\,e^{\imath\theta}+a_0}=\bar{0}$$

But $\bar{0}=0$, and the conjugate of a sum is equal to the sum of the conjugates, so the equation becomes:

$$\overline{a_n\,r^n\,e^{\imath n\theta}}+\overline{a_{n-1}\,r^{n-1}\,e^{\imath(n-1)\theta}}+\cdots+\overline{a_1\,r\,e^{\imath\theta}}+\overline{a_0}=0$$

Further, $\overline{kz}=k\bar{z}$, i.e. the conjugate of a complex number times by a constant is equal to the constant times by the conjugate of the complex number, so the equation becomes:

$$a_n\,r^n\,\overline{e^{\imath n\theta}}+a_{n-1}\,r^{n-1}\,\overline{e^{\imath(n-1)\theta}}+\cdots+a_1\,r\,\overline{e^{\imath\theta}}+a_0=0$$

But $\overline{e^{\imath x}}=\overline{\cos x+\imath\sin x}=\cos x-i\sin x=\cos(-x)+i\sin(-x)=e^{-ix}$ so:

$$a_n\,r^n\,e^{-in\theta}+a_{n-1}\,r^{n-1}\,e^{-i(n-1)\theta}+\cdots+a_1\,r\,e^{-i\theta}+a_0=0$$

$$a_n\left(r\,e^{-i\theta}\right)^n+a_{n-1}\left(r\,e^{-i\theta}\right)^{n-1}+\cdots+a_1\left(r\,e^{-i\theta}\right)+a_0=0$$

and therefore:

$$a_n\,(a-ib)^n+a_{n-1}\,(a-ib)^{n-1}+\cdots+a_1\,(a-ib)+a_0=0$$

$$\therefore P(a-ib)=0$$

which completes the proof.

If any of the polynomial's coefficients are not real, then the roots will NOT all occur in conjugate pairs.

Example 23

The polynomial $z^3-7z^2+25z-39$ has one zero equal to $2+3i$. Write its three linear factors.

Solution

The coefficients are real numbers and $2+3i$ is a zero, so from the conjugate root theorem $2-3i$ must be another zero.

$$z^3-7z^2+25z-39=(z-(2+3i))(z-(2-3i))Q(z)$$
$$=(z^2-4z+13)Q(z)$$ where $Q(z)$ is a 1st-degree polynomial.

Therefore you have $z^3-7z^2+25z-39=(z^2-4z+13)(z-k)$.

Equating the constant terms: $-39=13\times(-k)$

But $-39=13\times(-3)$, so $Q(z)=(z-3)$.

Hence $z^3-7z^2+25z-39=(z-(2+3i))(z-(2-3i))(z-3)$.

Multiple zeros of a polynomial

A polynomial of degree n has n zeros, but they are not necessarily all different. You say that c is a zero of multiplicity r when the factor $(z - c)$ occurs r times.

For example, if $P(z) = (z-1)^3(z-5)^2(z-6)$ then 1 is a zero of multiplicity three, 5 is a zero of multiplicity two and 6 is a zero of multiplicity one.

Furthermore, if $x = c$ is a zero of multiplicity r of the real polynomial $P(x)$, then $x = c$ is also a zero of multiplicity $(r-1)$ of the derived polynomial $P'(x)$, a zero of multiplicity $(r-2)$ of the second derived polynomial $P''(x)$, and so on:

If $\quad P(x) = (x-c)^r S(x) \quad$ where $r > 0,\ S(c) \neq 0$

then
$$\begin{aligned} P'(x) &= r(x-c)^{r-1}S(x) + (x-c)^r S'(x) \\ &= (x-c)^{r-1}\left[rS(x) + (x-c)S'(x)\right] \\ &= (x-c)^{r-1}Q(x) \quad [1] \end{aligned}$$

i.e. the polynomial $P'(x)$ has a zero $x = c$ of multiplicity $(r-1)$.

Applying the product rule to $P'(x)$ in [1] produces the polynomial $P''(x)$ with a zero $x = c$ of multiplicity $(r-2)$.

If $P(x)$ is a polynomial of degree n, then $P'(x)$ must be a polynomial of degree $(n-1)$, $P''(x)$ a polynomial of degree $(n-2)$ and so on. This property allows you to use calculus techniques to solve equations that are known to have multiple roots.

Example 24

Solve $z^4 - 6z^3 + 14z^2 - 30z + 45 = 0$ given that it has a real root of multiplicity 2.

Solution

Consider the polynomial $P(z) = z^4 - 6z^3 + 14z^2 - 30z + 45$

Differentiate: $P'(z) = 4z^3 - 18z^2 + 28z - 30$

The real root must be a factor of both 45 and 30, consider ±1, ±3, ±5 and substitute first into $P'(z)$.

$P'(1) = -16$, $P'(-1) = -80$, $P'(3) = 0$, $P'(-3) = -60$, $P'(5) = 160$, $P'(-5) = -1120$

Now find P(3): P(3) = 0

Hence $z = 3$ is the double real root.

$$P(z) = (z-3)^2(z^2 + bz + c)$$

Hence $9c = 45$ so $c = 5$ and $P(z) = (z-3)^2(z^2 + bz + 5)$

$(z^2 - 6z + 9)(z^2 + bz + 5) = z^4 - 6z^3 + 14z^2 - 30z + 45$

Equate coefficients of z: $-30 + 9b = -30$ so $b = 0$

Hence $P(z) = (z-3)^2(z^2 + 5)$

$$P(z) = 0:\ z = 3 \text{ or } z = \pm i\sqrt{5}$$