## Compound interest

In previous years you have used the compound interest formula in calculations:

$$A = P(1+r)^n$$

where $P$ is the principal (or starting amount)

$r$ is the interest rate for the period, expressed as a decimal

$n$ is the number of compounding periods

$A$ is the amount to which $P$ has grown or reduced (final amount)

This formula for compound interest may also be written as a recurrence relation.

If $P = A_0$

then $A_1 = A_0 \times R$

and $A_2 = A_1 \times R$

and eventually $A_n = A_{n-1} \times R$

where $A_n$ is the value of the initial amount after $n$ time periods and $R = 1 + \frac{r}{100}$, the interest rate being $r$% per time period.

When money is invested or borrowed, the compound interest formula is used to calculate what the investment or loan becomes over a period of time.

If payments are made to the investment or loan, the final amount changes, either increasing more rapidly in the case of an investment, or decreasing in the case of a loan.

### Example 1

Anh deposits \$1000 in an investment account that is paying a monthly interest rate of 0.25%, with the interest compounded monthly. Calculate the value of the investment after 12 months.

#### Solution

$P = 1000,\ r = 0.25\% = 0.0025,\ n = 12$

$A = P(1+r)^n$: $A = 1000 \times 1.0025^{12}$

$= 1030.42$

After 12 months, there is \$1030.42 in Anh's investment account.

### Example 2

Caleb knows that he will need \$10000 in five years time to pay for a study tour. He wishes to deposit enough in an investment account to achieve this goal. The account interest will be fixed at 3.6% per annum, compounded monthly. How much does he need to deposit into the account to achieve this goal? Round your answer to the next dollar.

#### Solution

$P = ?,\ n = 5 \times 12 = 60$ months, $A = 10000,\ r = \frac{3.6}{12} = 0.3\%$ per month $= 0.003$

Use $A = P(1+r)^n$: $10000 = P \times 1.003^{60}$

$$P = \frac{10000}{1.003^{60}}$$

$$P = 8354.95$$

Caleb should deposit \$8355.

## Example 3

Maria deposits \$1000 in an investment account that is paying a monthly interest rate of 0.25%, with the interest compounded monthly. She adds a further \$1000 to the account at the start of each subsequent month. How much is her investment worth at the end of six months?

## Solution

$P = 1000$, $r = 0.25\% = 0.0025$, $n = 6, 5, 4, 3, 2, 1$

The first \$1000 is invested for 6 months: $A = 1000 \times 1.0025^6 = 1015.09$

The second \$1000 is invested for 5 months: $A = 1000 \times 1.0025^5 = 1012.56$

The third \$1000 is invested for 4 months: $A = 1000 \times 1.0025^4 = 1010.04$

The fourth \$1000 is invested for 3 months: $A = 1000 \times 1.0025^3 = 1007.52$

The fifth \$1000 is invested for 2 months: $A = 1000 \times 1.0025^2 = 1005.01$

The sixth \$1000 is invested for 1 month: $A = 1000 \times 1.0025^1 = 1002.50$

The value of Maria's investment after 6 months = \$1015.09 + \$1012.56 + \$1010.04 + \$1007.52 + \$1005.01 + \$1002.50
= \$6052.72

This could have been found using the recurrence relation in a spreadsheet.

$A_0 = 1000$, $r = 0.0025$, $R = 1.0025$, $n = 6$

| | A | B | C | D |
|---|---|---|---|---|
| 1 | A0 | 1000.00 | | |
| 2 | r = | 0.0025 | | |
| 3 | R = | 1.0025 | | |
| 4 | | | | |
| 5 | n | An | Deposit | Total |
| 6 | 0 | 1000.00 | | 1000.00 |
| 7 | 1 | 1002.50 | 1000.00 | 2002.50 |
| 8 | 2 | 2007.51 | 1000.00 | 3007.51 |
| 9 | 3 | 3015.03 | 1000.00 | 4015.03 |
| 10 | 4 | 4025.06 | 1000.00 | 5025.06 |
| 11 | 5 | 5037.63 | 1000.00 | 6037.63 |
| 12 | 6 | 6052.72 | 0.00 | 6052.72 |

Later in this chapter you will use the sum of a geometric series to find the value of an investment like Maria's.

### Future value

In Example 1, the value of the investment, **Future value** (FV), could have been found from the **Present value** (PV), using the formula $FV = PV(1 + r)^n$, where $r$ is the interest rate per period, expressed as a decimal and $n$ is the number of compounding periods.

For Anh, PV = 1000, $r = 0.0025$ and $n = 12$: $FV = 1000 \times 1.0025^{12} = \$1030.42$.

This formula can be rearranged so that if you know the future value that you require then you can calculate the present value. The formula is $PV = \frac{FV}{(1+r)^n}$. The present value is the single amount, which if invested at the same rate for the same period, would give that future value.

If Anh wished to have \$1500 after 6 months at an interest rate of 0.25% per month, then this formula will give the amount that she has to invest.

FV = 1500, $r = 0.0025$, $n = 6$, find the PV: $PV = \frac{1500}{1.0025^6} = 1477.70$

Anh would need to make an initial deposit of \$1477.70.

> PV = present value, FV = future value
> $r$ = interest rate per period as a decimal
> $n$ is the number of compounding periods
> $FV = PV(1 + r)^n$
> $PV = \frac{FV}{(1+r)^n}$

## Annuities

An annuity is a compound interest investment from which equal payments are received on a regular basis (at equal periods of time) for a fixed period of time.

The payment is usually made at the end of the time period so that no interest is received until the end of the second time period.

## Future value

The future value of an investment or annuity is the total value of the investment at the end of the term of investment, including all contributions and interest earned.

## Present value

The present value of an investment or annuity is the single sum of money (or principal) that could be initially invested to produce a given future value over a given period of time.

The information needed to answer questions about annuities will be given in a table. The entries in this table have been calculated using the following formula.

$$\text{FVA} = a\left\{\frac{(1+r)^n - 1}{r}\right\} \text{ and } \text{PVA} = a\left\{\frac{(1+r)^n - 1}{r(1+r)^n}\right\}$$

where

FVA is the Future Value of an Annuity

PVA is the Present Value of an Annuity

$a$ is the contribution per period paid at the end of the period

$r$ is the interest rate per compounding period as a decimal

$n$ is the number of periods.

You can set up a spreadsheet using these formulae to see if your own calculations agree with the following for $a = 1$. This table gives the future value of an annuity of $1 at the given interest rate for the given period.

| Future value interest factors (FVA) | | | | | |
|---|---|---|---|---|---|
| **$1** | **Interest rate per period** | | | | |
| ***N*** | **1%** | **2%** | **3%** | **4%** | **5%** |
| 1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2 | 2.0100 | 2.0200 | 2.0300 | 2.0400 | 2.0500 |
| 3 | 3.0301 | 3.0604 | 3.0909 | 3.1216 | 3.1525 |
| 4 | 4.0604 | 4.1216 | 4.1836 | 4.2465 | 4.3101 |
| 5 | 5.0101 | 5.2040 | 5.3091 | 5.4163 | 5.5256 |
| 6 | 6.1520 | 6.3081 | 6.4684 | 6.6330 | 6.8019 |

This table gives the present value of an annuity of $1 at the given interest rate for the given period.

| Present value interest factors (PVA) | | | | | |
|---|---|---|---|---|---|
| $1 | Interest rate per period | | | | |
| *N* | **1%** | **2%** | **3%** | **4%** | **5%** |
| 1 | 0.9901 | 0.9804 | 0.9709 | 0.9615 | 0.9524 |
| 2 | 1.9704 | 1.9416 | 1.9135 | 1.8861 | 1.8594 |
| 3 | 2.9410 | 2.8839 | 2.8286 | 2.7751 | 2.7232 |
| 4 | 3.9020 | 3.8077 | 3.7171 | 3.6299 | 3.5460 |
| 5 | 4.8534 | 4.7135 | 4.5797 | 4.4518 | 4.3295 |
| 6 | 5.7955 | 5.6014 | 5.4172 | 5.2421 | 5.0757 |

The following examples show how to use these tables.

## Example 4

Use the table of future value interest factors to find:

**(a)** the future value of an annuity of $1000 per year for 4 years at 3% per annum

**(b)** the future value of an annuity of $1500 per year for 6 years at 4% per annum.

### Solution

**(a)** 4 years at 3% gives a multiplication factor of 4.1836 from the table.
FVA = $1000 × 4.1836 = $4183.60

**(b)** 6 years at 4% gives a multiplication factor of 6.6330 from the table.
FVA = $1000 × 6.6330 = $6633.00

## Example 5

Use the table of present value interest factors for an annuity of $1 per period.

**(a)** Jake plans to invest $5000 per year for 6 years in an annuity. His investment will earn interest at the rate of 4% per annum. Calculate the present value of this annuity.

**(b)** Ari takes out a personal loan of $9000 to be repaid over 4 years at an interest rate of 5% per year. Use the PVA table to find his yearly repayments.

### Solution

**(a)** 6 years at 4% gives a multiplication factor of 5.2421.
**PVA** = $5000 × 5.2421 = $26 210.50

**(b)** Let $M$ be the yearly repayments.
Using the PVA table with $r = 5\%$ and the period 4 years, the present value of $1 is 3.5460.
The formula used to create the table was PVA = $a$ × value in table, where $a$ = $1.
In this application, PVA = value of the loan and $a = M$, the repayments.
Hence $3.5460 \times M = 9000$

$$M = \frac{9000}{3.5460} = \$2538.07$$

Ari will have to repay $2538.05 every year for 4 years.

## Summary of present and future value interest factors

1 The table of future value interest factors is used to find the future value of an annuity for a given interest rate and a set time period. The table gives the future value of each dollar of the annuity. Multiply this value by the value of the annuity.

2 The table of present value interest factors is used to find the present (or current) value of an annuity for a given interest rate and a set time period. The table gives the present value of each dollar of the annuity. Multiply this value by the value of the annuity to obtain its present value.

3 The table of present value interest factors can be used to find the repayments required to pay off a loan of a given amount at a set rate of compound interest over a fixed time period. The repayments are obtained by dividing the value of the loan by the value of the cell in the table corresponding to the interest rate and time period.

### Example 6

The table gives the present value interest factors for an annuity of $1 for various interest rates, *r*, and number of periods, *N*.

| Present value interest factors | | | | | |
|---|---|---|---|---|---|
| $1 | Interest rate per period (as a decimal) (*r*) | | | | |
| *N* | **0.0025** | **0.005** | **0.0075** | **0.008** | **0.009** |
| 71 | 64.98140 | 59.64121 | 54.89293 | 54.00754 | 52.29657 |
| 72 | 65.81686 | 60.33951 | 55.47685 | 54.57097 | 52.82118 |
| 73 | 66.65023 | 61.03434 | 56.05643 | 55.12993 | 53.34111 |
| 74 | 67.48153 | 61.72571 | 56.63169 | 55.68446 | 53.85641 |
| 75 | 68.31075 | 62.41365 | 57.20267 | 56.23458 | 54.36710 |
| 76 | 69.13791 | 63.09815 | 57.76940 | 56.78034 | 54.87324 |

**(a)** Kumba plans to invest $300 each month for 76 months. Her investment will earn interest at a rate of 0.0025 (as a decimal) per month. Use the information in the table to calculate the present value of this annuity.

**(b)** Zephan uses the same table to calculate the loan repayments for his car loan. His loan is for $22 000 and will be repaid in equal monthly repayments over 6 years. The interest rate on his loan is 9.6% per annum. Calculate the amount of each monthly repayment, rounded to the next dollar.

### Solution

**(a)** 76 months at 0.0025 gives a multiplication factor of 69.13791 from the table.

Present value of the annuity $= \$300 \times 69.13791$

$= \$20\,741.37$

**(b)** 6 years $= 6 \times 12 = 72$ months, $N = 72$

9.6% p.a. = 0.8% per month so $r = 0.008$ as a decimal

From the table, the value to be used is 54.57097.

Hence $22\,000 = M \times 54.57097$

$$M = \frac{22\,000}{54.57097}$$

$$= 403.14$$

The repayments would be $404 per month, rounded to the next dollar.

## Effective annual rate of interest

The **effective annual rate** of interest is a way of comparing interest rates on a common basis even when they are quoted as per annum, per quarter, per month, per day etc. Without a common basis, it is very hard to compare the overall effects of different interest rates that are applied at different time periods.

By converting each interest rate to its effective annual rate, you can compare the costs of loans or the returns on investments so as to decide which one is best for you.

$$\text{Effective annual interest rate} = \left(1 + \frac{r}{n}\right)^n - 1$$

### Example 7

What is the effective annual interest rate for a loan advertised as:

**(a)** 6% p.a. compounded monthly

**(b)** 6% p.a. compounded quarterly

**(c)** 6% p.a. compounded daily (use 365 days in a year).

#### Solution

**(a)** $r = 0.06$, $n = 12$

$$\text{Effective annual interest rate} = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 6.168\%$$

**(b)** $r = 0.06$, $n = 4$

$$\text{Effective annual interest rate} = \left(1 + \frac{0.06}{4}\right)^{4} - 1 = 6.136\%$$

**(c)** $r = 0.06$, $n = 4365$

$$\text{Effective annual interest rate} = \left(1 + \frac{0.06}{365}\right)^{365} - 1 = 6.183\%$$

### Example 8

A home loan from Bank B is advertised with an interest rate of 5% p.a. compounded monthly. Credit Union C advertises their home loan with an interest rate of 4.9% p.a. compounded daily. If there are no other fees involved, which financial institution offers the better deal?

#### Solution

Bank B: $r = 0.05$, $n = 12$

$$\text{Effective annual interest rate} = \left(1 + \frac{0.05}{12}\right)^{12} - 1 = 5.116\%$$

Credit Union C: $r = 0.049$

$$\text{Effective annual interest rate} = \left(1 + \frac{0.049}{365}\right)^{365} - 1 = 5.022\%$$

Credit Union C offers the better deal.