INTEGRALS INVOLVING TRIGONOMETRIC SUBSTITUTION

If an expression of the form $a^2 - x^2$ occurs in the integrand, the standard trigonometric substitutions $x = a\sin\theta$ or $x = a\cos\theta$ will help to find the integral.

If an expression of the form $a^2 + x^2$ occurs in the integrand, then the standard substitution is $x = a\tan\theta$.

IMPORTANT NOTE: in this course, if these substitutions are needed, they should be given in the question.

Example 18

Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Solution

$x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; $\frac{dx}{d\theta} = 2\cos\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = |2\cos\theta| = 2\cos\theta$ because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\therefore \cos\theta \ge 0$

Limit for $x = 2$: $2 = 2\sin\theta$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

Limit for $x = 0$: $0 = 2\sin\theta$, $\sin\theta = 0$, $\theta = 0$

$$\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{2}} 2\cos\theta \times \frac{dx}{d\theta} \times d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\cos\theta \times 2\cos\theta\,d\theta$$

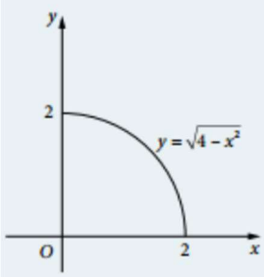
$$= 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= 2 \times \frac{\pi}{2} = \pi$$



Geometrically, the graph of $y = \sqrt{4-x^2}$ between $x = 0$ and $x = 2$ is the quarter of a circle with equation $x^2 + y^2 = 4$ (in the first quadrant). This integration thus proves the formula for the area of a circle: $A = \pi r^2$.

Area $= \frac{1}{4}\pi r^2 = \frac{1}{4} \times \pi \times 4 = \pi$ when $r = 2$.

Note: If the restriction on θ had been different, e.g. $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, then $\cos\theta \ge 0$ would not be true and you would not be able to use $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$. Instead, you may have needed $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = -2\cos\theta$.

Remember: Always take care with trigonometric substitutions and check the sign of the function for the given domain.

Example 19

Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using the substitution $x = \tan\theta$.

Solution

Method 1

$x = \tan\theta, \frac{dx}{d\theta} = \sec^2\theta$: $\quad \frac{1}{1+x^2} = \frac{1}{1+\tan^2\theta} = \frac{1}{\sec^2\theta}$

Limit for $x = 1$: $\quad 1 = \tan\theta, \theta = \frac{\pi}{4}$

Limit for $x = 0$: $\quad 0 = \tan\theta, \theta = 0$

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}dx &= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \frac{dx}{d\theta} \times d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \sec^2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{4}} d\theta \\ &= [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\end{aligned}$$

Method 2

$x = \tan\theta, dx = \sec^2\theta\, d\theta$

Limit for $x = 1$: $\quad 1 = \tan\theta, \theta = \frac{\pi}{4}$

Limit for $x = 0$: $\quad 0 = \tan\theta, \theta = 0$

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}dx &= \int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2\theta}\sec^2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \sec^2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{4}} d\theta \\ &= [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\end{aligned}$$

Example 20

Find $\int \frac{1-\sin x}{x+\cos x}dx$ using the substitution $u = x + \cos x$.

Solution

$u = x + \cos x, du = (1 - \sin x)\, dx$: $\quad$

$$\begin{aligned}\int \frac{1-\sin x}{x+\cos x}dx &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|x + \cos x| + C\end{aligned}$$

This example involves an indefinite integral, so you must remember to resubstitute for u at the end to make the answer a function of x.