RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

1 If α and β are roots of the equation $x^2 + 8x - 5 = 0$, find the quadratic equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

The quadratic equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ factorises as $k\left(x-\dfrac{\alpha}{\beta}\right)\left(x-\dfrac{\beta}{\alpha}\right)$ (where k is any number).

Expanding $\left(x-\dfrac{\alpha}{\beta}\right)\left(x-\dfrac{\beta}{\alpha}\right)$, we obtain: $x^2 - x\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right) + \dfrac{\alpha}{\beta}\times\dfrac{\beta}{\alpha} = x^2 - x\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right) + 1$

Now: $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha} = \dfrac{\alpha^2}{\alpha\beta}+\dfrac{\beta^2}{\alpha\beta} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

But $\alpha+\beta = -\dfrac{b}{a} = \dfrac{-8}{1} = -8$ and $\alpha\beta = \dfrac{c}{a} = \dfrac{-5}{1} = -5$

So $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha} = \dfrac{(-8)^2-2\times(-5)}{(-5)} = \dfrac{64+10}{-5} = -\dfrac{74}{5}$

Therefore, the quadratic equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ is $k\left[x^2+\dfrac{74}{5}x+1\right]$

2 If α and β are roots of the equation $x^2 + 4x + 1 = 0$, find the value of:

(a) $\alpha+\beta$ **(b)** $\alpha\beta$ **(c)** $\alpha^2+\beta^2$ **(d)** $\alpha^3+\beta^3$

a) $\alpha+\beta = -\dfrac{b}{a} = \dfrac{-4}{1} = -4$

b) $\alpha\beta = \dfrac{c}{a} = \dfrac{1}{1} = 1$

c) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (-4)^2 - 2\times 1 = 16-2 = 14$

d) $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= (-4)^3 - 3\times 1\times(-4)$

$= -52$

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4 If α and β are roots of the equation $px^2 + qx + r = 0$, find the following in terms of p, q and r.

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(b)** $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

a) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$ But $\alpha + \beta = -\frac{q}{p}$ and $\alpha\beta = \frac{r}{p}$

So $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{-q/p}{r/p} = -\frac{q}{r}$

b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$\underline{\qquad} = \frac{(-q/p)^2 - 2 \times r/p}{(r/p)^2} = \frac{+q^2/p^2 - \frac{2r}{p}}{\frac{r^2}{p^2}} = \frac{q^2 - 2pr}{r^2}$

6 Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$, given that the sum of two of its roots is zero.

Three roots α, β, γ

$\alpha + \beta + \gamma = -\frac{b}{a} = \frac{3}{1} = 3$ But $\alpha + \beta = 0$ so $\gamma = 3$

$\therefore$ we can factorise by $(x-3)$

$x^3 - 3x^2 - 4x + 12 = (x-3)(x^2 - 4)$

$\underline{\qquad} = (x-3)(x-2)(x+2).$

The two other roots (except $\gamma = 3$) are

$\alpha = 2$ and $\beta = -2$

whose sum is zero indeed.

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9 Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$, given that the product of two of the roots is 4.

There are 3 roots α, β, γ

$\alpha\beta\gamma = \frac{-d}{a} = -\frac{12}{3} = -4$ But $\alpha\beta = 4$ so $\gamma = -1$

we can therefore factorise by $(x-(-1)) = (x+1)$

$3x^3 - 17x^2 - 8x + 12 = (x+1)(3x^2 - 20x + 12)$

$\Delta = (20)^2 - 4 \times 12 \times 3 = 256 = 16^2$

$x_1 = \frac{20+16}{6} = 6$ $x_2 = \frac{20-16}{6} = \frac{4}{6} = \frac{2}{3}$

So there are 3 solutions: $\alpha = 6$, $\beta = \frac{2}{3}$ and $\gamma = -1$

11 Find two values of m, such that the roots of the equation $x^3 + 2x^2 + mx - 16 = 0$ are $\alpha, \beta, \alpha\beta$. Using these values of m, find α and β.

$\alpha\beta(\alpha\beta) = -\frac{d}{a} = \frac{16}{1} = 16$ so $(\alpha\beta)^2 = 16$ so $\alpha\beta = \pm 4$

Further $\alpha\beta + \alpha(\alpha\beta) + \beta(\alpha\beta) = \frac{c}{a} = m$

So $\alpha\beta + \alpha\beta(\alpha + \beta) = m$

But $\alpha + \beta + \alpha\beta = -\frac{b}{a} = \frac{-2}{1} = -2$ so $\alpha + \beta = -2 - \alpha\beta$.

Case 1: if $\alpha\beta = 4$ then $\alpha + \beta = -2 - 4 = -6$

and $m = 4 + 4 \times (-6) = -20$

Case 2 if $\alpha\beta = -4$ then $\alpha + \beta = -2 + 4 = 2$

and $m = -4 - \cancel{4} \times 2 = -12$ So $m = -12$ or $m = -20$

if $m = -20$ then $\alpha + \beta = -6$ (so $\beta = -6 - \alpha$) and $\alpha\beta = 4$

so $\alpha(-6-\alpha) = 4$, i.e. $-\alpha^2 - 6\alpha - 4 = 0$ $\Delta = 36 - 16 = 20$

$\alpha = \frac{6 + \sqrt{20}}{-2} = -3 + \sqrt{5}$ and $\beta = -3 - \sqrt{5}$

if $m = -12$ then $\alpha + \beta = 2$ (so $\beta = 2 - \alpha$) and $\alpha\beta = -4$

so $\alpha(2-\alpha) = -4$ or $-\alpha^2 + 2\alpha + 4 = 0$ $\Delta = 4 + 16 = 20$

$\alpha = \frac{-2 + \sqrt{20}}{-2} = 1 - \sqrt{5}$ and $\beta = 2 - (1 - \sqrt{5}) = 1 + \sqrt{5}$

15 Solve the equation $4x^3 - 12x^2 + 9x - 2 = 0$, given that two of its roots are equal.

There are 3 roots α, β and γ Say $\alpha = \beta$.

* $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{12}{4} = 3$ so $2\alpha + \gamma = 3$ ①

* $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{9}{4}$ so $\alpha^2 + \alpha\gamma + \alpha\gamma = \frac{9}{4}$

or $\alpha(\alpha + 2\gamma) = \frac{9}{4}$ ②

* $\alpha\beta\gamma = -\frac{d}{a} = \frac{2}{4} = \frac{1}{2}$ so $\alpha^2\gamma = \frac{1}{2}$ ③

From ① $\gamma = 3 - 2\alpha$

Substituting in ② $\alpha[\alpha + 2(3 - 2\alpha)] = \frac{9}{4}$

so $-3\alpha^2 + 6\alpha = \frac{9}{4}$ so $-12\alpha^2 + 24\alpha - 9 = 0$

$4\alpha^2 - 8\alpha + 3 = 0$

$\Delta = 64 - 4 \times 3 \times 4 = 16 = 4^2$

so $\alpha = \frac{8+4}{8} = \frac{3}{2}$ or $\alpha = \frac{8-4}{8} = \frac{1}{2}$

Then $\gamma = 3 - 2\alpha$ or $\gamma = 3 - 2 \times \frac{1}{2} = 2$

$\gamma = 3 - 2 \times \frac{3}{2} = 0$

Not possible. as 0 is not a root.

So $4x^3 - 12x^2 + 9x - 2 = (x-2)(4x^2 - 4x + 1)$

$= 4(x-2)\left(x^2 - x + \frac{1}{4}\right)$

$= 4(x-2)\left(x - \frac{1}{2}\right)^2$

3 roots are 0.5, 0.5 and 2

19 Solve the equation $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, given that the sum of two of its roots is zero.

There are 4 roots $\alpha, \beta, \gamma, \delta$ and $\alpha + \beta = 0$ (so $\beta = -\alpha$)

$* \ \alpha + \beta + \gamma + \gamma = -\frac{b}{a} = \frac{2}{8} = \frac{1}{4}$ so $\gamma + \delta = \frac{1}{4}$ $\delta = \frac{1}{4} - \gamma$

$* \ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = -\frac{6}{8} = -\frac{3}{4}$

so $\alpha\beta\underbrace{(\gamma + \delta)}_{= \frac{1}{4}} + \gamma\delta\underbrace{(\alpha + \beta)}_{= 0} = -\frac{3}{4}$

so $\frac{\alpha\beta}{4} = -\frac{3}{4}$ so $\alpha\beta = -3$

but $\alpha + \beta = 0$ $(\beta = -\alpha)$

so $-\alpha^2 = -3$ $\alpha^2 = 3$ so $\alpha = \pm\sqrt{3}$

and $\beta = \mp\sqrt{3}$.

<u>Then</u>: $\alpha\beta\gamma\delta = \frac{e}{a} = \frac{9}{8}$ But $\alpha\beta = -3$

so $-3\gamma\delta = \frac{9}{8}$ or $\gamma\delta = -3/8$

But $\gamma + \delta = \frac{1}{4}$ so, to find γ and δ, we need

to solve the quadratic equation $x^2 - \frac{1}{4}x - \frac{3}{8} = 0$

or $8x^2 - 2x - 3 = 0$ $\Delta = 4 - 4 \times (-3) \times 8 = 100 = 10^2$

Two roots $\gamma = \frac{2 + 10}{16} = \frac{12}{16} = \frac{3}{4}$ and $\delta = \frac{2 - 10}{16} = -\frac{1}{2}$

The 4 roots are $\pm\sqrt{3}$, $\frac{3}{4}$ and $\left(-\frac{1}{2}\right)$.

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20 If α, β, γ are the roots of $3x^3 + 8x^2 - 1 = 0$, find the value of: $\left(\beta+\frac{1}{\gamma}\right)\left(\gamma+\frac{1}{\alpha}\right)\left(\alpha+\frac{1}{\beta}\right)$.

We expand:

$$\left(\beta+\frac{1}{\gamma}\right)\left(\gamma+\frac{1}{\alpha}\right)\left(\alpha+\frac{1}{\beta}\right) = \left(\beta+\frac{1}{\gamma}\right)\left(\alpha\gamma+\frac{\gamma}{\beta}+1+\frac{1}{\alpha\beta}\right)$$

$$= \alpha\beta\gamma + \gamma + \beta + \frac{1}{\alpha} + \alpha + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\alpha\beta\gamma}$$

$$= \alpha\beta\gamma + (\alpha+\beta+\gamma) + \frac{1}{\alpha\beta\gamma} + \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)$$

But $\alpha+\beta+\gamma = -\frac{b}{a} = \frac{-8}{3}$

and $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{0}{3} = 0$

and $\alpha\beta\gamma = -\frac{d}{a} = \frac{1}{3}$

So:

$$= \frac{1}{3} - \frac{8}{3} + 3 + \left(\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}\right)$$

$$= \frac{2}{3} + \frac{0}{1/3}$$

$$= \frac{2}{3}$$