TANGENTS AND NORMALS TO A CURVE

Differentiation gives the gradient function of a curve. This function can be evaluated at a point on the curve to obtain the gradient of the curve's **tangent** at that point. The point-gradient form of the straight line can then be used to find the equation of that tangent.

A **normal** at a point on a curve is the line that is perpendicular to the tangent at that point. Because the lines are perpendicular, the gradient of the normal can be obtained from the gradient of the tangent at that point.

Example 26

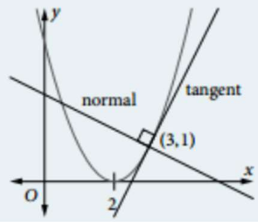
Find the equation of the tangent and the normal to the curve $y = x^2 - 4x + 4$ at the point on the curve where $x = 3$.

Solution

$x = 3$, $y = 9 - 12 + 4 = 1$ so the point is $(3, 1)$

$y = x^2 - 4x + 4$: $\quad \dfrac{dy}{dx} = 2x - 4$

$x = 3$: $\quad \dfrac{dy}{dx} = 6 - 4 = 2$



Equation of the tangent (from point-gradient form): $\quad y - 1 = 2(x - 3)$

$$y = 2x - 5$$

Gradient of normal $= \dfrac{-1}{\text{gradient of tangent}} = -\dfrac{1}{2}$ $\quad$ (Remember $m_1 \times m_2 = -1$ for perpendicular lines)

Equation of the normal (from point-gradient form): $\quad y - 1 = -\frac{1}{2}(x - 3$

$$2y - 2 = -x + 3$$

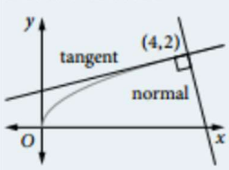
$$x + 2y - 5 = 0$$

Example 27

Find the equation of the tangent and normal to the curve $y = \sqrt{x}$ at the point on the curve where $x = 4$.

Solution

Draw a sketch.



Point is at $x = 4$, $y = 2$: $\quad y = x^{\frac{1}{2}}$: $\quad \dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

$x = 4$: $\quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

For $(4, 2)$, $m = \frac{1}{4}$, tangent is: $\quad y - 2 = \frac{1}{4}(x - 4)$

$$4y - 8 = x - 4$$

The equation of the tangent is $x - 4y + 4 = 0$.

For $(4, 2)$, $m = -4$, normal is: $\quad y - 2 = -4(x - 4)$

The equation of the normal is $4x + y - 18 = 0$.

Example 28

Find the equation of the tangent to the curve $y = \frac{x^3}{3} - x^2 - x + 1$ at the points on the curve where the tangent is parallel to the line $7x - y + 5 = 0$.

Solution

$y = \frac{x^3}{3} - x^2 - x + 1$: $\quad \frac{dy}{dx} = x^2 - 2x - 1$

Gradient of $7x - y + 5 = 0$: $\quad m = 7$

Solve: $\quad x^2 - 2x - 1 = 7$

$$x^2 - 2x - 8 = 0$$
$$(x-4)(x+2) = 0$$
$$x = -2, 4$$

At $x = 2$, $y = -\frac{11}{3}$; at $x = 4$, $y = \frac{7}{3}$.

At $\left(-2, -\frac{11}{3}\right)$, tangent is: $\quad y + \frac{11}{3} = 7(x+2)$

$$3y + 11 = 21x + 42$$
$$21x - 3y + 31 = 0$$

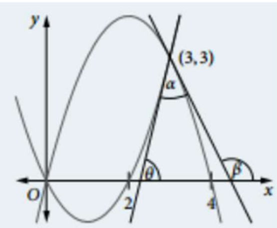
At $\left(4, \frac{7}{3}\right)$, tangent is: $\quad y - \frac{7}{3} = 7(x-4)$

$$3y - 7 = 21x - 84$$
$$21x - 3y - 77 = 0$$

Example 29

Find the points of intersection of the parabolas $y = x^2 - 2x$ and $y = 4x - x^2$. Sketch the parabolas and find the angle between the parabolas at their point of intersection in the first quadrant.

Solution



$y = x^2 - 2x$, $y = 4x - x^2$, intersection at: $\quad x^2 - 2x = 4x - x^2$

$$2x^2 - 6x = 0$$
$$2x(x-3) = 0$$
$$x = 0, 3 \quad y = 0, 3$$

Curves intersect at $(0,0)$ and $(3,3)$. Intersection point in the first quadrant is $(3,3)$. The angle between the curves at this point is the angle between their tangents at this point.

For $y = x^2 - 2x$: $\quad \frac{dy}{dx} = 2x - 2$

At $(3,3)$: $\quad \frac{dy}{dx} = 6 - 2 = 4 \qquad$ Hence $\tan\theta = 4$, so $\theta = 75°58'$

For $y = 4x - x^2$: $\quad \frac{dy}{dx} = 4 - 2x$

At $(3,3)$: $\quad \frac{dy}{dx} = 4 - 6 = -2 \qquad$ Hence $\tan\beta = -2$, so $\beta = 180° - 63°26' = 116°34'$

Now $\beta = \alpha + \theta$ (exterior angle of a triangle), so $\alpha = 116°34' - 75°58' = 40°36'$.