COMBINATIONS

Simplify:

${}^{7}C_2 = \frac{7!}{2!(7-2)!} = 21$	${}^{9}C_1 = 9$	${}^{8}C_4 = 70$	${}^{12}C_{10} = 66$
$\binom{12}{4} = 495$	${}^{n}C_n = \frac{n!}{0!\,n!} = 1$ So ${}^{n}C_n = 1$	${}^{n}C_1 = \frac{n!}{1!(n-1)!}$ $= n$ So ${}^{n}C_1 = n$	${}^{n}C_{n-1} = \frac{n!}{(n-1)!(n-(n-1))!}$ $= n$ So ${}^{n}C_{n-1} = n$

3 In how many ways can three books be selected from eight different books?

${}^{8}C_3 = 56$

4 How many different hands of five cards can be dealt from a standard pack of 52 playing cards?

${}^{52}C_5 = 2,598,960$

6 The number of ways of picking six numbers from 45 numbers is:

$\boxed{\textbf{A}}$ ${}^{45}C_6$ **B** ${}^{45}P_6$ **C** $6 \times {}^{45}C_1$ **D** $6 \times {}^{45}P_1$

7 In how many ways can a set of two cooks and three waiters be selected from five cooks and four waiters?

${}^{5}C_2 \times {}^{4}C_3 = 40$

9 In how many ways can a committee of four teachers and five parents be formed from eight teachers and seven parents?

${}^{8}C_4 \times {}^{7}C_5 = 1,470$

10 From eight lawyers, seven clerks and five judges, how many different groups could be formed that contain five lawyers, four clerks and three judges? Indicate whether each answer is correct or incorrect.

(a) ${}^{20}C_{12}$ **(b)** ${}^{8}P_5 \times {}^{7}P_4 \times {}^{5}P_3$ $\boxed{\textbf{(c)}}$ ${}^{8}C_5 \times {}^{7}C_4 \times {}^{5}C_3$ **(d)** 19600

11 In how many ways can eight different rabbits be divided into a group of five and a group of three?

${}^{8}C_5 = 56$

12 A committee of six is selected from ten people, of whom *A* and *B* are two. How many committees can be formed:

(a) containing both *A* and *B*

(b) excluding *A* if *B* is included?

a) ${}^{8}C_4 = 70$

b) ① if B included, then A is excluded, so ${}^{8}C_5$

② if B not included, then either | A included ${}^{8}C_5$

OR | A excluded ${}^{8}C_6$

TOTAL ${}^{8}C_5 + {}^{8}C_5 + {}^{8}C_6 = 140$

13 In how many ways can three cards be selected from a standard pack of 52 playing cards if:

(a) at least one selected card is an Ace

(b) no more than one selected card is an Ace?

a) ${}^{52}C_3 - {}^{48}C_3 = 4,804$

Alternative method $4 \times {}^{48}C_2 + {}^{4}C_2 \times {}^{48}C_1 + 4 = 4512 + 288 + 4 = 4804$

b) ${}^{48}C_3 + 4 \times {}^{48}C_2 = 21,808$

14 In how many ways can a team of three runners and four hurdlers be chosen from six runners and seven hurdlers?

${}^{6}C_3 \times {}^{7}C_4 = 700$

15 A committee of seven politicians is chosen from ten Liberal members, eight Labor members and five independents. In how many ways can this be done to include exactly one independent, at least three Liberal members and at least one Labor member?

5 choices for the independent, then either:

① 3 LIB and 3 LAB : ${}^{10}C_3 \times {}^{8}C_3 = 6,720$

OR ② 4 LIB and 2 LAB : ${}^{10}C_4 \times {}^{8}C_2 = 5,880$

OR ③ 5 LIB and 1 LAB : ${}^{10}C_5 \times {}^{8}C_1 = 2,016$

TOTAL 14,616

we times by 5 to allow for the independent.

So $N = 5 \times 14,616 = 73,080$

16 A team of 11 is chosen from 15 cricketers. Five of the 15 cricketers are bowlers only, two are wicketkeepers only and the rest are batters only. How many possible teams can be chosen that contain:

(a) four bowlers, one wicketkeeper and six batters

(b) at least four bowlers and at least one wicketkeeper?

a) ${}^5C_4 \times {}^2C_1 \times {}^8C_6 = 280$

b) 1 wicketkeeper, 4 bowlers: ${}^2C_1 \times {}^5C_4 \times {}^8C_6 = 280$

1 wicket keeper, 5 bowlers: ${}^2C_1 \times {}^5C_5 \times {}^8C_5 = 112$

2 wicketkeepers, 4 bowlers: ${}^2C_2 \times {}^5C_4 \times {}^8C_5 = 280$

2 wicket keepers, 5 bowlers: ${}^2C_2 \times {}^5C_5 \times {}^8C_4 = 70$

TOTAL 742

17 From seven teachers and five students, a committee of seven is formed. How many different committees can be selected if teachers and students are both represented and the teachers are in a majority?

There could be: 4 teachers: ${}^7C_4 \times {}^5C_3 = 350$

OR 5 teachers ${}^7C_5 \times {}^5C_2 = 210$

OR 6 teachers ${}^7C_6 \times {}^5C_1 = 35$

TOTAL 595

18 From four oranges, three bananas and two apples, how many selections of five pieces of fruit can be made, taking at least one of each kind?

EITHER: 2 apples, 1 banana, 2 oranges: ${}^2C_2 \times {}^3C_1 \times {}^4C_2 = 18$

OR 2 apples, 2 bananas, 1 orange: ${}^2C_2 \times {}^3C_2 \times {}^4C_1 = 12$

OR 1 apple, 2 bananas, 2 orange: ${}^2C_1 \times {}^3C_2 \times {}^4C_2 = 36$

OR 1 apple, 1 banana, 3 oranges: ${}^2C_1 \times {}^3C_1 \times {}^4C_3 = 24$

OR 1 apple, 3 bananas, 1 orange: ${}^2C_1 \times {}^3C_3 \times {}^4C_1 = 8$

TOTAL 98

COMBINATIONS

12 10 7

20 In how many ways can a jury of twelve people be chosen from ten women and seven men so that there are at least six women and not more than four men?

EITHER 4M, 8W $^7C_4 \times {}^{10}C_8 = 1,575$

OR 3M, 9W $^7C_3 \times {}^{10}C_9 = 350$

OR 2M, 10W $^7C_2 \times {}^{10}C_{10} = 21$

TOTAL 1,946

21 In how many ways can a group of three or more be selected from nine?

$$^9C_3 + {}^9C_4 + {}^9C_5 + {}^9C_6 + {}^9C_7 + {}^9C_8 + {}^9C_9$$
$$= 84 + 126 + 126 + 84 + 36 + 9 + 1$$
$$= 466$$

3W 4G 7W 6G

22 In how many ways can a committee of three women and four girls be chosen from seven women and six girls so that if the eldest woman is serving on the committee then the youngest girl is not?

If the eldest woman is on the committee $^6C_2 \times {}^5C_4 = 75$

If the eldest woman is NOT on the committee $^6C_3 \times {}^6C_4 = 300$

TOTAL 375

3 C 2 V

23 How many **(a)** selections or **(b)** arrangements, consisting of three consonants and two vowels, can be made from eight different consonants and four different vowels?

8 C 4 V

a) $^8C_3 \times {}^4C_2 = 336$

b) $336 \times 5! = 40,320$

the order matters.

4 P 3 M

24 In how many ways can four Physics books and three Mathematics books be arranged on a shelf if a selection is made from six different Physics books and five different Mathematics books? In how many of these arrangements are all the Physics books next to each other?

6 P 5 M

a) $^6C_4 \times {}^5C_3 \times 7! = 756,000$

the order matters

b) $^6C_4 \times {}^5C_3 \times 4! \times 4 \times 3! = 86,400$

arrangement of 4 Physics books

Position of 1st Physics book

Arrangement of Maths books

COMBINATIONS

25 In how many ways can three glasses and two plates be arranged in a row if a selection is made from five different glasses and four different plates? In how many of these arrangements does a glass occupy the middle position?

3G 2P 5G 4P

a) $^5C_3 \times {}^4C_2 \times \underbrace{5!}_{\text{order matters}} = 7,200$

b) $^5C_3 \times {}^4C_2 \times \underbrace{3}_{\text{3 glasses}} \times \underbrace{4!}_{\text{order matters}} = 4,320$

26 How many words (i.e. any arrangements of letters) containing three consonants and two vowels can be formed from the letters of the word PROMISE?

$$^4C_3 \times {}^3C_2 \times 5! = 1,440$$

27 If, $^nC_6 = {}^nC_4$, find the value of n.

We know that $^nC_r = {}^nC_{n-r}$

so $r = 6$ and $n - r = 4$

$\therefore\ n - 6 = 4$ so $n = 10$

28 In how many ways can nine books be distributed among a teacher, a parent and a child, if the teacher receives four, the parent three and the child two?

$$^9C_4 \times {}^5C_3 \times {}^2C_2 = 1,260$$

29 In how many ways can eight different toys be divided into two unequal groups?

either : 7 and 1 $\quad ^8C_7 = 8$

OR 6 and 2 $\quad ^8C_6 = 28$

OR 5 and 3 $\quad ^8C_5 = 56$

TOTAL = 92

30 In how many ways can eight basketball players be divided into four groups of two?

$$\left[^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2\right] \div \underbrace{4!}_{\text{No order for the 4 groups}} = 105$$

31 In how many ways can n objects be shared between two people? ('Shared' means that each person gets at least one.)

Each object can be included or excluded.

$$\text{So } \underbrace{2 \times 2 \times 2 \ldots \times 2}_{n \text{ times}} - 1 - 1 = 2^n - 2$$

A gets no object

B gets no object

32 From the definition of ${}^nC_r$, prove each of the following:

a) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

$$\text{LHS} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r+1)!\,[n-(r+1)]!} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r+1)!\,(n-r-1)}$$

$$\text{LHS} = \frac{n!\,(r+1)}{(r+1)!\,(n-r)!} + \frac{n!\,(n-r)}{(r+1)!\,(n-r)!}$$

$$\text{LHS} = \frac{n!\,(r+1) + n!\,(n-r)}{(r+1)!\,(n-r)!}$$

$$\text{LHS} = \frac{n!\,[r+1+n-r]}{(r+1)!\,(n-r)!}$$

$$\text{LHS} = \frac{n!\,(n+1)}{(r+1)!\,(n-r)!}$$

$$\text{LHS} = \frac{(n+1)!}{(r+1)!\,[n+1-r-1]!}$$

$$\text{LHS} = \frac{(n+1)!}{(r+1)!\,[(n+1)-(r+1)]!} = {}^{n+1}C_{r+1}$$

32 From the definition of ${}^nC_r$, prove each of the following:

b) ${}^nC_r + 2 \times {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$

$$LHS = \frac{n!}{(n-r)!\,r!} + 2\frac{n!}{(r-1)!\,[n-(r-1)]!} + \frac{n!}{(r-2)!\,[n-(r-2)]!}$$

$$LHS = \frac{n!}{(n-r)!\,r!} + \frac{2\,n!\,r}{r!\,[n-r+1]!} + \frac{n!\,r(r-1)}{r!\,[n-r+2]!}$$

$$LHS = \frac{n!\,(n-r+1)(n-r+2)}{[n-r+2]!\,r!} + \frac{2\,n!\,r\,(n-r+2)}{r!\,[n-r+2]!} + \frac{n!\,r(r-1)}{r!\,[n-r+2]!}$$

$$LHS = \frac{n!}{[n-r+2]!\,r!}\Big[(n-r+1)(n-r+2) + 2r(n-r+2) + r(r-1)\Big]$$

$$LHS = \frac{n!}{[n-r+2]!\,r!}\Big[n^2 \cancel{-nr} + 2n \cancel{-rn} + \cancel{r^2} \cancel{-2r} + n \cancel{-r} + 2 + \cancel{2rn} \cancel{-2r^2} + \cancel{4r} + \cancel{r^2} \cancel{-r}\Big]$$

$$LHS = \frac{n!}{[(n+2)-r]!\,r!}\left[n^2+3n+2\right]$$

$$LHS = \frac{n!}{[(n+2)-r]!\,r!}\left[(n+1)(n+2)\right]$$

$$LHS = \frac{(n+2)!}{[(n+2)-r]!\,r!} = {}^{n+2}C_r$$

33 The ratio of the number of combinations of $(2n + 2)$ different objects taken n at a time to the number of combinations of $(2n - 2)$ different objects taken n at a time is 99 : 7. Find the value of n.

$$\frac{{}^{2n+2}C_n}{{}^{2n-2}C_n} = \frac{\dfrac{(2n+2)!}{n!\,[(2n+2)-n]!}}{\dfrac{(2n-2)!}{n!\,[(2n-2)-n]!}} = \frac{\dfrac{(2n+2)!}{[n+2]!}}{\dfrac{(2n-2)!}{[n-2]!}}$$

$$\frac{{}^{2n+2}C_n}{{}^{2n-2}C_n} = \left(\frac{(2n+2)!}{[n+2]!}\right) \div \left(\frac{(2n-2)!}{(n-2)!}\right) = \frac{(2n+2)!}{(n+2)!} \times \frac{(n-2)!}{(2n-2)!}$$

$$\frac{{}^{2n+2}C_n}{{}^{2n-2}C_n} = \frac{(2n+2)(2n+1)(2n)(2n-1)}{(n+2)(n+1)(n)(n-1)}$$

$$\frac{{}^{2n+2}C_n}{{}^{2n-2}C_n} = \frac{2\cancel{(n+1)}(2n+1) \times 2\cancel{n} \times (2n-1)}{(n+2)\cancel{(n+1)} \times \cancel{n} \times (n-1)}$$

$$\frac{{}^{2n+2}C_n}{{}^{2n-2}C_n} = \frac{4(2n+1)(2n-1)}{(n+2)(n-1)} = \frac{99}{7}$$

So $4(2n+1)(2n-1) \times 7 = 99 \times (n+2)(n-1)$

$\iff 4 \times 7 \times (4n^2-1) = 99(n^2+n-2)$

$\iff 13n^2 - 99n + 170 = 0$ quadratic equation

$\Delta = 99^2 - 4 \times 170 \times 13 = 961 = 31^2$ so two roots.

$n_1 = \dfrac{99-31}{26} = \dfrac{34}{13}$ impossible as it's not an integer

$n_2 = \dfrac{99+31}{26} = 5$ ONLY possible value

So $\boxed{n=5}$