# HARDER EXPONENTIAL GROWTH AND DECAY

$\frac{dN(t)}{dt} = k[N(t) - P]$ means that the rate of change of $N$ is proportional to the excess of $N$ over a fixed quantity $P$. This can be applied to several real-life processes.

First note that the notation $N(t)$ , which shows that the function $N(t)$ depends of t, is often abbreviated N. So $\frac{dN(t)}{dt} = k[N(t) - P]$ becomes in abbreviation notation $\frac{dN}{dt} = k(N - P)$

However we must remember that in this equation, the term $N$ represents a function, whereas the terms $P$ and $k$ represent constants.

The general solution of the differential equation $\frac{dN}{dt} = k(N - P)$ is $N(t) = P + A\,e^{kt}$

**Proof:**

If $N(t) = P + A\,e^{kt}$ then $\frac{dN(t)}{dt} = A \times k\,e^{kt}$

as $\frac{d(e^x)}{dx} = e^x$, and using the Chain rule for $e^{kt}$

whereas $k(N - P) = k(P + A\,e^{kt} - P) = A \times k\,e^{kt}$

therefore indeed $N(t) = P + A\,e^{kt}$ is a solution of the differential equation $\frac{dN}{dt} = k(N - P)$

It can be shown that this family of functions are the only solutions of this differential equation (beyond the range of this course).

## Example 9

$N$ is increasing according to the equation $\frac{dN}{dt} = 0.4(N - 50)$. If $N = 60$ when $t = 0$:

**(a)** show that $N = 50 + Ae^{0.4t}$ is a solution to this equation, where $A$ is a constant

**(b)** calculate the value of $N$ when $t = 20$.

## Solution

**(a)** Differentiate $N = 50 + Ae^{0.4t}$: $\frac{dN}{dt} = 0.4Ae^{0.4t}$

Rewrite $N = 50 + Ae^{0.4t}$: $Ae^{0.4t} = N - 50$

Substitute into $\frac{dN}{dt} = 0.4Ae^{0.4t}$: $\frac{dN}{dt} = 0.4(N - 50)$

Hence $N = 50 + Ae^{0.4t}$ is a solution to the equation $\frac{dN}{dt} = 0.4(N - 50)$.

**(b)** At $t = 0, N = 60$: $60 = 50 + A$

$A = 10$

$\therefore$ $N = 50 + 10e^{0.4t}$

At $t = 20$: $N = 50 + 10e^{8}$

$N \approx 29\,860$

## Example 10

The mass $M$ of a particular southern right whale is modelled as $M = 55 - 54e^{-kt}$, where $M$ is measured in tonnes, $t$ is the age of the whale in years and $k$ is a positive constant.

**(a)** Show that the rate of growth of the whale's mass is given by the differential equation $\frac{dM}{dt} = k(55 - M)$.

**(b)** What is the birth mass of the whale? (i.e. at $t = 0$)

**(c)** When the whale is one year old, its mass is 10 tonnes. Show that $k = \ln\left(\frac{6}{5}\right)$.

**(d)** What is the mass of the whale when it is 10 years old (to the nearest tonne)?

**(e)** If male southern right whales grow to about 55 tonnes and females grow to about 85 tonnes, determine the gender of this whale, giving reasons for your answer.

## Solution

**(a)** $M = 55 - 54e^{-kt}$

Differentiate with respect to $t$: $\frac{dM}{dt} = 54ke^{-skt}$

Rewrite $M = 55 - 54e^{-kt}$: $54e^{-kt} = 55 - M$

Substitute into $\frac{dM}{dt} = 54ke^{-kt}$: $\frac{dM}{dt} = k(55 - M)$

Hence the rate of growth of the whale's mass is $\frac{dM}{dt} = k(55 - M)$.

**(b)** At $t = 0$: $M = 55 - 54e^0$

$M = 55 - 54 = 1$

The birth mass of the whale is 1 tonne.

**(c)** At $t = 1$, $M = 10$: $10 = 55 - 54e^{-k}$

$54e^{-k} = 45$

$e^{-k} = \frac{45}{54} = \frac{5}{6}$

$e^{k} = \frac{6}{5}$

$k = \ln\left(\frac{6}{5}\right)$

**(d)** At $t = 10$: $M = 55 - 54e^{-10\ln\left(\frac{6}{5}\right)}$

$M = 55 - 54e^{10\ln\left(\frac{5}{6}\right)}$

$M \approx 46.3$

The mass of the whale to the nearest tonne is 46 tonnes.

**(e)** As $t \to \infty$: $M \to 55 - 54e^{-\infty}$

$M \to 55$

The limiting mass of the whale is 55 tonnes, so it is most likely to be a male.

# HARDER EXPONENTIAL GROWTH AND DECAY

**Newton's law of cooling**

Newton's law of cooling states that the cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium.

$$\frac{dT}{dt} = -k(T - M)$$

where $T$ is the temperature at any time $t$ and $M$ is the temperature of the surrounding medium (a constant)

## Example 11

The original temperature of a body is 100°C, the temperature of its surroundings is 20°C and the body cools to 70°C in 10 minutes. Assuming Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T-20)$, where $T$ is the temperature of the body at time $t$, find:

**(a)** the temperature of the body after 20 minutes **(b)** the time taken to cool from 100°C to 60°C.

### Solution

$$\frac{dT}{dt} = -k(T-20)$$

Reciprocal of both sides: $\frac{dt}{dT} = \frac{-1}{k(T-20)}, \ T \neq 20$

Integrate with respect to $T$: $t = -\frac{1}{k}\int \frac{1}{T-20}\,dT$

$$t = -\frac{1}{k}\log_e(T-20) + C, \ T > 20$$

$$-k(t-C) = \log_e(T-20)$$

$$T - 20 = e^{-k(t-C)}$$

Let $A = e^{-kC}$ (a constant): $T = 20 + Ae^{-kt}$

When $t = 0$, $T = 100$: $100 = 20 + A$

$$A = 80$$

When $t = 10$, $T = 70$: $70 = 20 + 80e^{-10k}$

$$e^{-10k} = \frac{50}{80} = 0.625$$

$$-10k = \log_e 0.625$$

$$-10k \approx -0.47$$

$k = 0.047$ (using the approximate logarithm value)

$$\therefore \ T = 20 + 80e^{-0.047t}$$

**(a)** When $t = 20$: $T = 20 + 80e^{-0.94}$

$$T \approx 51.25$$

After 20 minutes the temperature is approximately 51.25°C.

**(b)** When $T = 60$: $60 = 20 + 80e^{-0.047t}$

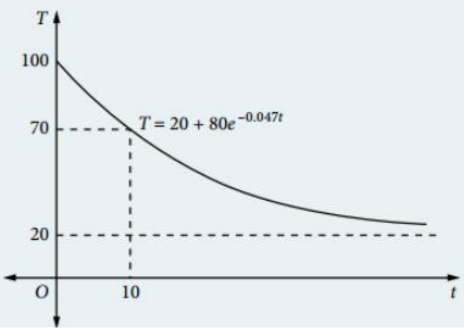
$$e^{-0.047t} = \frac{40}{80} = 0.5$$

$$-0.047t = \log_e 0.5$$

$$-0.047t \approx -0.6931$$

$$t \approx 14.7$$

The temperature reaches 60°C after approximately 14.7 minutes.




The graph of $T = 20 + 80e^{-0.047t}$ shows that the temperature of the body never falls below the temperature of the surroundings. As $t \to \infty$, $T \to 20$ from above.

# HARDER EXPONENTIAL GROWTH AND DECAY

**Wilhelmy's law**

Many chemical reactions follow a law that states that the rate of the reaction is proportional to the difference between the initial concentration of the reagent (i.e. the chemical reacting) and the amount transformed at any time.

$\frac{dx}{dt} = k(a - x)$ where $a$ is the initial concentration and $x$ is the amount transformed at time $t$

**Example 12**

A chemical reaction follows the rule $\frac{dx}{dt} = k(a - x)$, where $a$ is the initial concentration and $x$ is the amount of the reagent transformed at time $t$. Thus when $t = 0$, $x = 0$. If $a = 10$ and after 2 minutes $x = 4$, find the concentration of the reagent after 5 minutes.

**Solution**

$$\frac{dx}{dt} = k(10 - x),\ 0 \le x \le 10$$

Reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{k(10-x)},\ 0 \le x < 10$

Integrate with respect to $x$: $t = \frac{1}{k}\int \frac{1}{10-x}\,dx$

$$kt = -\log_e(10 - x) + C$$

When $t = 0$, $x = 0$: $0 = -\log_e 10 + C$

$$C = \log_e 10$$

$$\therefore\ kt = \log_e 10 - \log_e(10 - x)$$

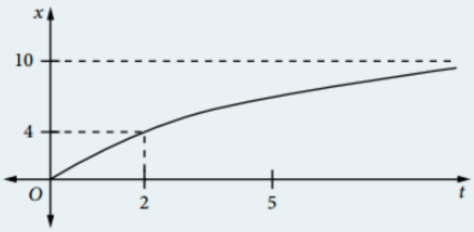
$$kt = \log_e \frac{10}{10-x}$$

Use inverse functions: $\frac{10}{10-x} = e^{kt}$

$$\frac{10-x}{10} = e^{-kt}$$

$$10 - x = 10e^{-kt}$$

$$x = 10(1 - e^{-kt}) \text{ for } t \ge 0$$




Note that the asymptote of the graph is $x = 10$, approached from below. This is consistent with the restriction $0 \le x < 10$ for $t \ge 0$.

When $t = 2$, $x = 4$: $4 = 10(1 - e^{-2k})$

$$0.4 = 1 - e^{-2k}$$

$$e^{-2k} = 0.6$$

Reciprocal of both sides: $e^{2k} = \frac{5}{3}$

$$2k = \log_e\left(\frac{5}{3}\right)$$

$$k = \frac{1}{2}\log_e\left(\frac{5}{3}\right)$$

When $t = 5$: $x = 10(1 - e^{-5k})$

But: $e^{-2k} = 0.6$

$$e^{-5k} = \left(e^{-2k}\right)^{\frac{5}{2}} = \left(\frac{3}{5}\right)^{\frac{5}{2}} \approx 0.279$$

$$\therefore\ x = 10(1 - 0.279)$$

$$x = 10 - 2.79$$

$$10 - x = 2.79$$

**or** inverse functions: $-2k = \log_e 0.6$

$$-2k = -0.51$$

$$k = 0.255$$

i.e. $x = 10(1 - e^{-5k})$

$$x = 10(1 - e^{-1.275})$$

$$x = 10(1 - 0.279)$$

$$x = 10 - 2.79$$

$$10 - x = 2.79$$

Hence the concentration is 2.79 units after 5 minutes. (Remember that $(10 - x)$ is the concentration remaining.)

Note that given $\frac{dN}{dt} = k(N - P)$, where $k$ and $P$ are constants, if $k < 0$, then $\lim_{t\to+\infty} N = P$