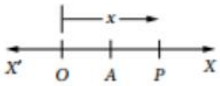
VELOCITY AND ACCELERATION AS RATES OF CHANGE

Particle is the term used for a body that behaves as if all forces acting on the body are acting through a single point. This means that the body can be represented as a single point, regardless of its actual size and shape. This definition of a particle means that quite large bodies, e.g. trains, can still be classified as 'particles' provided this condition applies.

Displacement



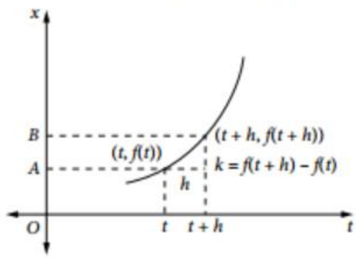
Consider a particle, represented by a point P, moving in a straight line $X'OX$.

The **displacement** x is the particle's position relative to the fixed point O. It may be a positive or negative number, according to whether P is to the right or left of O. The origin of the motion is not necessarily at O, so when $t = 0$, P may be (for example) at the point A.

Displacement is defined as the position relative to a starting point. It can be positive or negative.
Displacement is not necessarily the total distance travelled.
Unlike displacement, **distance** is always a positive quantity.

Velocity

Consider the equation $x = f(t)$, which gives the position coordinate x of a particle moving in a straight line at time t.



At time t, the particle is at A, and at time $(t + h)$ the particle is at B, as shown in the diagram. Thus in the small time interval h the particle has changed its position by an amount $k = f(t + h) - f(t)$.

The average **velocity** in this time interval $= \dfrac{k}{h} = \dfrac{f(t+h) - f(t)}{h}$, $h \neq 0$.

The instantaneous velocity of the particle at time t is defined by $\lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$. This may be denoted by $v(t)$, $f'(t)$, $\dfrac{dx}{dt}$ or $\dot{x}$:

$$v(t) = f'(t) = \frac{dx}{dt} = \dot{x} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Velocity is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction.
Velocity can be positive or negative, depending on the direction of travel.
Speed is the magnitude of the velocity and is always positive.

Acceleration

Acceleration is defined as the rate of change of velocity with respect to time.
Acceleration, like velocity, can be positive or negative.
Positive acceleration indicates that the velocity is increasing, while negative acceleration indicates that the velocity is decreasing (which is often called deceleration or retardation).

(*Note:* 'increasing velocity' is not necessarily 'faster speed'; it only means acceleration in the direction of positive displacement.)

If velocity is denoted by $v(t)$, then the average acceleration over the time interval from t to $(t + h)$ is $\dfrac{v(t+h) - v(t)}{h}$.

The instantaneous acceleration at time t is defined by $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$. This may be denoted by $v'(t)$, $a(t)$, $f''(t)$, $\dfrac{dv}{dt}$, $\dfrac{d^2x}{dt^2}$ or $\ddot{x}$:

$$a(t) = v'(t) = \frac{d^2x}{dt^2} = \ddot{x} = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

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Summary of important motion terms

'initially':	$t = 0$	'at the origin':	$x = 0$
'at rest':	$v = 0$	'velocity is constant':	$a = 0$

Units and symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in Physics)
Velocity	cm s^{-1}, m s^{-1}	$v, \frac{dx}{dt}, \dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	$a, \frac{dv}{dt}, \frac{d^2x}{dt^2}, \ddot{x}$

Note that 's' is the abbreviation for second, 'cm' for centimetre and 'm' for metre.
Constant acceleration due to gravity on Earth can be assumed to be $9.8\,\text{m s}^{-2}$ ($\approx 10\,\text{m s}^{-2}$).

Example 3

A particle is moving in a straight line, so that its displacement x metres from a fixed point O on the line at time t seconds ($t \geq 0$) is given by $x = t^3 - 2t^2 - 4t$.

(a) Determine expressions for the velocity and acceleration of the particle.

(b) Find when the particle is at rest.

(c) Find when the acceleration is negative.

(d) By drawing the graph of the velocity function, comment on the velocity when the acceleration is negative.

(e) When is the acceleration positive? What does this mean for the velocity of the particle?

Solution

(a) $x = t^3 - 2t^2 - 4t$

Differentiate: $\frac{dx}{dt} = 3t^2 - 4t - 4$

$\therefore v = 3t^2 - 4t - 4$

Differentiate again: $\frac{dv}{dt} = 6t - 4$

$\therefore a = 6t - 4$

(b) At rest when $v = 0$: $3t^2 - 4t - 4 = 0$

$(3t + 2)(t - 2) = 0$

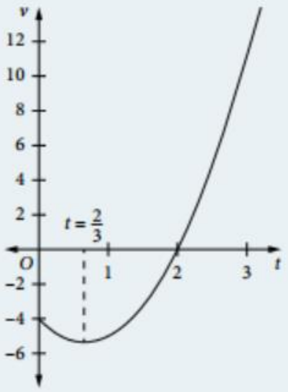
$t = -\frac{2}{3}, 2$

As $t \geq 0$, the particle is at rest after 2 seconds.

(c) $a < 0$: $6t - 4 < 0$

$t < \frac{2}{3}$

(d)



When $t < \frac{2}{3}$ the velocity function is a decreasing function and the velocity is negative. It obtains its least value when $t = \frac{2}{3}$.

(e) The acceleration is positive when $t > \frac{2}{3}$. This means the velocity is increasing when $t > \frac{2}{3}$, and since the graph of the velocity is getting steeper it means the velocity is increasing at an increasing rate.