INTRODUCTION TO DIFFERENTIAL EQUATIONS

The study of differential equations began in the late 1600s with Sir Isaac Newton's investigation of the orbits of the planets about the Sun. Newton referred to these equations as 'fluxional equations'. The term **differential equation** was suggested by Newton's contemporary, Gottfried Leibniz, who did much of the early work on them.

A differential equation is an equation that relates some unknown differentiable function to one or more of its derivatives.

For example, the general form of the first-order differential equation $y'(t) = f(t, y(t))$ expresses the rate of change of a quantity $y(t)$ in terms of two variables: the time t and the value of the quantity $y(t)$ itself.

Differential equations are a powerful way to represent, understand and predict the behaviour of variable quantities, including systems that change with time.

Whereas the solution of an algebraic equation such as $x + 1 = 0$ is a number, the solution of a differential equation is a function. More specifically, the solution of a differential equation will be a differentiable function $y = g(x)$ if the differential equation is true when y and its derivatives are replaced with $g(x)$ and its derivatives.

A solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is any differentiable function $y = g(x)$ with a derivative $\frac{dy}{dx} = g'(x)$ so that $g'(x) = f(x, g(x))$ for all $x \in (a, b)$.

In other words, substituting the solution $y = g(x)$ into the differential equation will reduce $\frac{dy}{dx} = f(x, y)$ to an identity in the independent variable x, for all values of this variable in some open interval of the x-axis. Therefore:

To verify that a function $y = g(x)$ is a solution of a differential equation, you can substitute the function and its derivative(s) into both sides of the differential equation and check that both sides are identically equal.

Verification of a solution to a differential equation

Example 1

Verify by differentiation that $y = 3e^{-x}$ is a solution of the differential equation $\frac{dy}{dx} = -y$.

Solution

Calculate the LHS of the equation: $\text{LHS} = \frac{dy}{dx}$

$= 3\left(-e^{-x}\right)$

$= -3e^{-x}$

Calculate the RHS of the equation: $\text{RHS} = -y$

$= -3e^{-x}$

The LHS of the equation is identically equal to the RHS of the equation for all relevant values of the independent variable.

Graphing particular members of a general solution

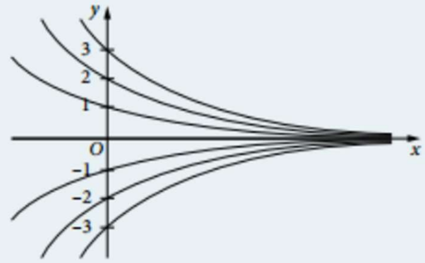
Example 2

Graph the following members of the one-parameter general solution $y = Ae^{-x}$ with $A \in \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution

The graph is drawn for $A = -3$, i.e., $y = -3e^{-x}$.

Next, separate graphs are drawn on the same axes for the other given values of A: $y = -2e^{-x}$, $y = -3e^{-x}$, etc. This should give the set of graphs shown.



A **particular solution** (or **solution curve**) of a differential equation is a unique function that is found by giving specific values to the parameters in the general solution. The parameters in the solution are chosen so that the particular solution satisfies one or more extra requirements called **initial conditions** or **initial values**. Initial conditions are also sometimes called boundary values.

Every particular solution of a first-order differential equation is the unique solution of an appropriate **initial value problem**.

An initial value problem of a first-order differential equation: $\begin{cases} y' = f(t, y(t)) \\ y(a) = y_a \\ t \in [a, b] \end{cases}$

consists of the differential equation together with its initial condition, requiring you to determine a particular solution $y(t)$ over a specific interval of the independent variable $a \le t \le b$.

Finding the particular solution to satisfy an initial condition

Example 3

Find the particular member of the general solution $y = Ae^{-x}$, where A is a real number that passes through the point with coordinates (0, 3).

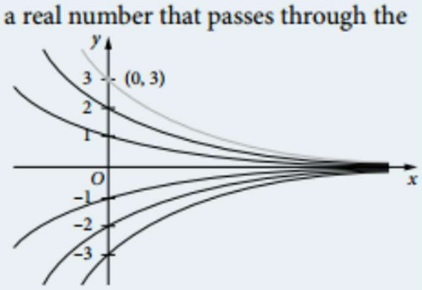
Solution

$y = Ae^{-x}$

To satisfy the relevant initial condition, substitute the given values.

$x = 0, y = 3$: $3 = Ae^{0}$

$\therefore A = 3$



The particular solution curve that passes through the point (0, 3) is identified.

Finding missing parameter(s) in a trial solution

A trial solution to a differential equation is a general solution with unspecified parameters that is tested to see if it satisfies an initial condition.

Example 4

Verify by differentiation that $y = ae^{\sin x}$ is a solution of the differential equation $\frac{dy}{dx} = y\cos x,\ y(0) = 2$, for a suitable choice of the parameter a.

Solution

The required derivatives of the trial solution are calculated to prove that LHS = RHS:

$$\begin{aligned} \text{LHS} &= \frac{dy}{dx} \\ &= \frac{d}{dx}\left(ae^{\sin x}\right) \\ &= \cos x\left(ae^{\sin x}\right) \\ &= y\cos x \\ &= \text{RHS} \end{aligned}$$

Check if the initial condition is satisfied, $y(0) = 2$: $y(0) = ae^{\sin 0}$

$2 = a$

$\therefore\ y = 2e^{\sin 0}$ is a particular solution.

Example 5

Verify by differentiation that $x = Ae^{st}$ is a solution of $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$, $x(0) = 1$ and $x'(0) = 2$, for suitable values of s.

Solution

Calculate any required derivatives of the trial solution $x = Ae^{st}$:

$$\frac{dx}{dt} = sAe^{st}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{d}{dt}\left(sAe^{st}\right)$$

$$= s^2Ae^{st}$$

Substitute into the LHS of the equation given, $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$:

$$\text{LHS} = s^2Ae^{st} - 5sAe^{st} + 6Ae^{st}$$

$$= Ae^{st}\left(s^2 - 5s + 6\right)$$

$$= 0$$

Solve for possible values of the parameter s: $Ae^{st}(s^2 - 5s + 6) = 0$

As $Ae^{st} \neq 0, t \in R$, so: $(s^2 - 5s + 6) = 0$

$(s - 2)(s - 3) = 0$

$\therefore s = 2$ or $s = 3$

The solution could be $x = Ae^{2t}$ or $x = Ae^{3t}$.

Check initial conditions: $x(0) = 1, x = Ae^{st} \therefore x(0) = Ae^0 = 1 \therefore A = 1$

$x'(0) = 2, x' = sAe^{st} = se^{st} \therefore se^0 = 2 \therefore s = 2$

Hence $s = 2$ is the only solution that satisfies both the differential equation and the initial conditions.

$\therefore x(t) = e^{2t}$ is a particular solution.

The number of parameters in the general solution of a differential equation, which determines the number of initial conditions required to fix these parameters, depends on the **order** and the **degree** of the differential equation.

The order of a differential equation is equal to the highest order derivative of the dependent variable.

The degree of a differential equation is the highest power of that highest order derivative.

In the nth-order differential equation $\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right)$, the dependent variable (y) always appears in the numerator of any derivatives and the independent variable x appears in the denominator.

A **first-order first-degree differential equation** for the unknown dependent variable y is an equation that involves only the first derivative of y. All first-order first-degree differential equations for y can be expressed in the form $\frac{dy}{dx} = f(x, y)$ for a suitable choice of the function $f(x, y)$ of the independent variable x and dependent variable y. The particular solution of a first-order first-degree differential equation requires a single **initial condition** $(x, y) = (x_0, y_0)$.

Classifying differential equations

Example 6

Classify the following differential equations according to their order and degree.

(a) $\frac{d\theta}{dt} = k(1+0.2\cos\theta)^2$ **(b)** $\left(\frac{dw}{dz}\right)^2 = 4\cos^2 z$ **(c)** $\frac{d^2x}{dt^2} - c\frac{dx}{dt} + kx = F$

Solution

(a) This equation defines a first-order first-degree differential equation because it involves only a first-order derivative to a power of one.

Variable θ is in the numerator of $\frac{d\theta}{dt}$, so it is the dependent variable, while t is in the denominator, so it is the independent variable.

(b) This equation defines a first-order second-degree differential equation because it involves only a first-order derivative to a power of two.

Variable w is in the numerator of $\frac{dw}{dz}$, so it is the dependent variable, while z is in the denominator, so it is the independent variable.

(c) This equation defines a second-order first-degree differential equation because it involves a second-order derivative $\frac{d^2x}{dt^2}$ of the dependent variable to the power of one.

Variable x is in the numerator of $\frac{d^2x}{dt^2}$, so it is the dependent variable, while t is in the denominator, so it is the independent variable.

Note: In each example considered so far, the solution is given in explicit form, such as $y = g(x)$.