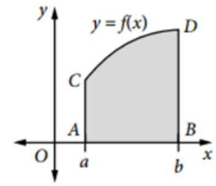
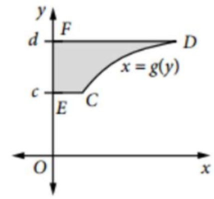
AREA BOUNDED BY THE Y-AXIS

The techniques used so far can also be used to find areas where one of the boundaries is the y-axis. You can effectively swap the roles of the x and y variables.



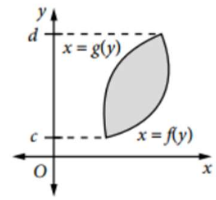
The area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b f(x)\,dx \qquad \text{or} \qquad \text{Area} = \int_a^b y\,dx$$



Similarly, the area bounded by the curve $x = g(y)$, the y-axis and the abscissae (i.e. horizontal lines) at $y = c$ and $y = d$ is given by:

$$\text{Area} = \int_c^d g(y)\,dy \qquad \text{or} \qquad \text{Area} = \int_c^d x\,dy$$



For two curves $x = f(y)$ and $x = g(y)$ that intersect at $y = c$ and $y = d$, where $f(y) \geq g(y)$ over the interval $c \leq y \leq d$, the area bounded by the curves is given by:

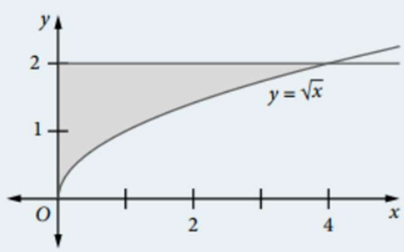
$$\text{Area} = \int_c^d \big(f(y) - g(y)\big)\,dy$$

Example 18

Calculate the area of the region bounded by the curve $y = \sqrt{x}$, the y-axis and the line $y = 2$.

Solution

Sketch the region:



Write the equation as a function of y (i.e. make x the subject):

$$y = \sqrt{x} \rightarrow \sqrt{x} = y$$

$$x = y^2$$

$$\text{Area} = \int_0^2 x\,dy = \int_0^2 y^2\,dy$$

$$= \left[\frac{y^3}{3}\right]_0^2$$

$$= \frac{8}{3} - 0$$

$$= 2\frac{2}{3}$$

Area of the region is $2\frac{2}{3}$ units^2.

Example 19

Calculate the area of the region bounded by the curve $y = \sqrt{x}$, the line $y = 6 - x$ and the y-axis.

Solution

Find the point of intersection: $\sqrt{x} = 6 - x$

Square both sides: $(\sqrt{x})^2 = (6 - x)^2$

$$x = 36 - 12x + x^2$$

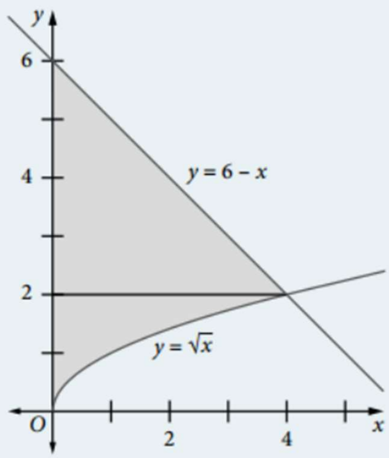
$$x^2 - 13x + 36 = 0$$

$$(x - 4)(x - 9) = 0$$

$$x = 4, 9$$

Only $x = 4$ is a root of $\sqrt{x} = 6 - x$: intersection point is $(4, 2)$.

Sketch the region:



Write the equations as functions of y:

$$y = \sqrt{x} \rightarrow y^2 = x$$

$$y = 6 - x \rightarrow x = 6 - y$$

$$\text{Area} = \int_0^2 y^2\,dy + \int_2^6 (6 - y)\,dy$$

$$= \left[\frac{y^3}{3}\right]_0^2 + \left[6y - \frac{y^2}{2}\right]_2^6$$

$$= \left(\frac{8}{3} - 0\right) + (36 - 18) - (12 - 2)$$

$$= 10\frac{2}{3}$$

Area of the region is $10\frac{2}{3}$ units^2.