SOLUTION SET OF SIMULTANEOUS EQUATIONS

2 Find algebraically the coordinates of the intersection points of:

(a) the straight line $y = x - 3$ and the circle $x^2 + y^2 = 9$

(b) the straight line $y = 2x - 1$ and the parabola $y = x^2 - 3x + 5$

(c) the straight line $y = 3 - 2x$ and the parabola $y = (x - 2)^2$

a) $\begin{cases} y = x - 3 \\ x^2 + y^2 = 9 \end{cases} \Leftrightarrow \begin{cases} y = x - 3 & \text{Equation } ① \\ x^2 + (x-3)^2 = 9 & ② \end{cases}$

$② \Leftrightarrow 2x^2 - 6x + 9 = 9 \Leftrightarrow 2x^2 - 6x = 0 \Leftrightarrow x^2 - 3x = 0$

$\Leftrightarrow x(x-3) = 0$ so either $x = 0$ and therefore $y = -3$

or $x = 3$ and therefore $y = 0$

So 2 points $(3,0)$ and $(0,-3)$

b) $\begin{cases} y = 2x - 1 \\ y = x^2 - 3x + 5 \end{cases} \Leftrightarrow \begin{cases} y = 2x - 1 \\ 2x - 1 = x^2 - 3x + 5 & ② \end{cases}$

$② \Rightarrow x^2 - 5x + 6 = 0 \qquad \Delta = 25 - 4 \times 6 = 1$

So either $x_1 = \dfrac{5-1}{2} = 2$ (and in that case $y = 3$) point $(2,3)$

or $x_2 = \dfrac{5+1}{2} = 3$ (and in that case $y = 5$) point $(3,5)$

c) $\begin{cases} y = 3 - 2x \\ y = (x-2)^2 \end{cases} \Leftrightarrow \begin{cases} y = 3 - 2x \\ (3 - 2x) = x^2 - 4x + 4 & ② \end{cases}$

$② \Leftrightarrow x^2 - 2x + 1 = 0 \Leftrightarrow (x-1)^2 = 0$

So $x = 1$ and therefore $y = 3 - 2 \times 1 = 1$

One point $(1,1)$ of intersection.

SOLUTION SET OF SIMULTANEOUS EQUATIONS

3 For what value of c is the line $y = 2x + c$ a tangent to the parabola $y = x^2 - x - 2$?

A $c = -4.25$ B $c = 0.25$ C $c = -2.25$ D $c = 4.25$

The line is tangent to the parabola if they intersect at only 1 point

$2x + c = x^2 - x - 2 \iff x^2 - 3x - 2 - c = 0$

$\Delta = (-3)^2 - 4 \times 1 \times (-2-c) = 9 + 8 + 4c = 17 + 4c$

$\Delta = 0$ when $c = -17/4$

For this value of c, $\Delta = 0$, and $\therefore$ the 2 graphs touch at only one point. Response $\boxed{A}$

4 For what value of c is the line $y = x + c$ a tangent to the circle $x^2 + y^2 = 4$?

The two curves are tangent if they intersect at only one point, ie when the discriminant of the equation $x^2 + (x+c)^2 = 4$ is equal to 0.

$\iff 2x^2 + 2xc + c^2 - 4 = 0$

$\Delta = (2c)^2 - 4 \times 2 \times (c^2 - 4) = -4c^2 + 32$

$\Delta = 0$ when $-4c^2 + 32 = 0$ i.e. $c^2 = \frac{32}{4}$ i.e $c = \pm\sqrt{8}$

For those 2 values, $\sqrt{8}$ and $-\sqrt{8}$, $\Delta = 0$, and the 2 curves are tangent to each other.

SOLUTION SET OF SIMULTANEOUS EQUATIONS

7 For what value of m does the line $y = mx - 6$ **(a)** touch **(b)** intersect **(c)** not intersect the parabola $y = x^2 - 2x + 3$?

$mx - 6 = x^2 - 2x + 3 \iff x^2 - (m+2)x + 9 = 0$

$\Delta = (m+2)^2 - 4 \times 9 \times 1 = m^2 + 4m - 32$

We look for the values of m for which $\Delta = 0$

$\Delta_2 = 4^2 - 4 \times (-32) = 144 = 12^2 \quad m_1 = \frac{-4-12}{2} = -8$

$m_2 = \frac{-4+12}{2} = 4 \quad \therefore$ for $m = -8$ or $m = 4$, $\Delta = 0$ and the two curves touch each other.

When $m < -8$ or $m > 4$ $\Delta > 0$ there are 2 roots, so the curves intersect.

When $-8 < m < 4$ $\Delta < 0$, there are no roots, the curves do not intersect.

11 For what value of a does the line $y = ax$ not meet the rectangular hyperbola $y = \frac{3}{x-2}$?

We look for the values for which it meets.

$ax = \frac{3}{x-2} \iff ax^2 - 2ax - 3 = 0$

$\Delta = (-2a)^2 - 4 \times a \times (-3) = 4a^2 + 12a = 4a(a+3).$

So $\Delta = 0$ when $a = 0$ or $a = -3$

outside those values $\Delta > 0$, so the 2 curves intersect

So the 2 curves do not meet when $-3 < a < 0$