The result $\frac{dy}{dx} \times \frac{dx}{dy} = 1$

Proof

Let $y = f(x)$. Differentiate both sides with respect to y:

$$\frac{d}{dy}(y) = \frac{d}{dy}(f(x))$$

$$\therefore \quad 1 = \frac{d}{dx}(f(x)) \times \frac{dx}{dy} \qquad \text{(using chain rule on RHS)}$$

$$\therefore \quad 1 = \frac{dy}{dx} \times \frac{dx}{dy}$$

Hence $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

Demonstration

Let $y = x^3 - 1 \qquad \therefore \qquad x = (y+1)^{\frac{1}{3}}$

$$\frac{dy}{dx} = 3x^2 \qquad \frac{dx}{dy} = \frac{1}{3}(y+1)^{-\frac{2}{3}}$$

$$\frac{dy}{dx} \times \frac{dx}{dy} = 3x^2 \times \frac{1}{3}(y+1)^{-\frac{2}{3}}$$

$$= \frac{x^2}{(y+1)^{\frac{2}{3}}}$$

$$= \frac{x^2}{(x^3)^{\frac{2}{3}}}$$

$$= \frac{x^2}{x^2} = 1$$

Derivative of $\sin^{-1} x$

$\sin^{-1} x$ is defined for $-1 \le x \le 1$.

Let $y = \sin^{-1} x \quad \therefore \quad x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

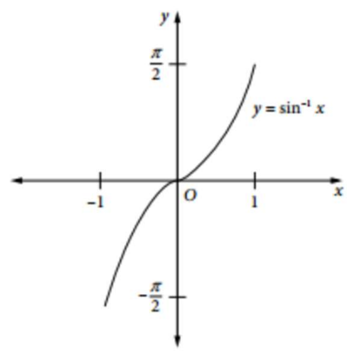
$$\frac{dx}{dy} = \cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} \text{ noting that } \cos y \ne 0 \therefore -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Now, using $\cos^2 y + \sin^2 y = 1$: $\quad \frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}} \qquad$ Which one?

As y is an angle in the first or fourth quadrants, $\cos y$ must be positive.

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1$$



Note that the derivative is not defined at $x = \pm 1$ (the graph of $\sin^{-1} x$ has vertical tangents at its endpoints).

Also note that the derivative is positive for all x in its domain (as $\sin^{-1} x$ is an increasing function).

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

Derivative of $\sin^{-1} \frac{x}{a}$

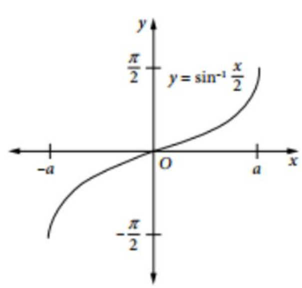
$\sin^{-1} \frac{x}{a}$ is defined for $-a \le x \le a$.

Let $y = \sin^{-1} \frac{x}{a} \quad \therefore \quad x = a \sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\frac{dx}{dy} = a\cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{a\cos y} \qquad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\text{i.e.} \quad \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}} \qquad \text{for } -a < x < a$$



$$\frac{d}{dx}\left(\sin^{-1} \frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}, -a < x < a$$

Derivative of $\cos^{-1}x$

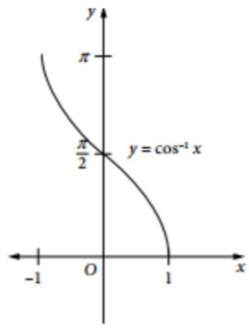
$\cos^{-1}x$ is defined for $-1 \le x \le 1$.

Let $y = \cos^{-1}x \quad \therefore \quad x = \cos y,\ 0 \le y \le \pi$

$$\frac{dx}{dy} = -\sin y$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sin y} \quad \text{for } 0 < y < \pi$$

$$\text{i.e. } \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1$$



$$\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$$

As for the derivative of $\sin^{-1}x$, the last step again uses $\cos^2 y + \sin^2 y = 1$.

Note that the gradient is negative for all x in the domain.

Note also that the domain of the derived function is different to the domain of $\cos^{-1}x$. Why?

Derivative of $\cos^{-1}\frac{x}{a}$

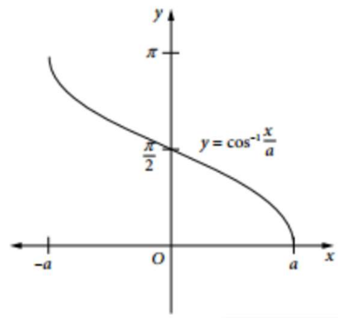
$\cos^{-1}\frac{x}{a}$ is defined for $-a \le x \le a$.

Let $y = \cos^{-1}\frac{x}{a} \quad \therefore \quad x = a\cos y,\ 0 \le y \le \pi$

$$\frac{dx}{dy} = -a\sin y$$

$$\therefore \frac{dy}{dx} = -\frac{1}{a\sin y} \quad \text{for } 0 < y < \pi$$

$$\text{i.e. } \frac{dy}{dx} = -\frac{1}{\sqrt{a^2-x^2}} \quad \text{for } -a < x < a$$



$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2-x^2}},\ -a < x < a$$

Example 21

Find the derivative of $\cos^{-1}(2x+1)$, stating the values of x for which it is defined.

Solution

$\cos^{-1}(2x+1)$ is defined for $-1 \le 2x+1 \le 1$

i.e. for $-2 \le 2x \le 0$

$-1 \le x \le 0$

Hence $\cos^{-1}(2x+1)$ is defined for $-1 \le x \le 0$.

Let $y = \cos^{-1}(2x+1)$

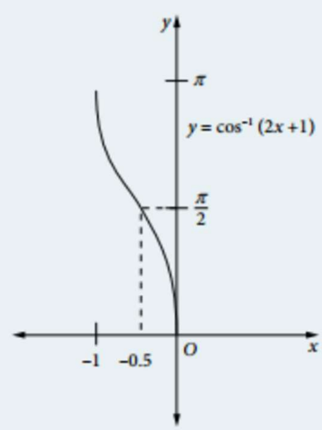
$= \cos^{-1}u \quad$ where $u = 2x+1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -\frac{1}{\sqrt{1-u^2}} \times 2$$

$$= -\frac{2}{\sqrt{1-(2x+1)^2}} \quad \text{provided } -1 < x < 0$$

$$= -\frac{2}{\sqrt{-4x(x+1)}}$$

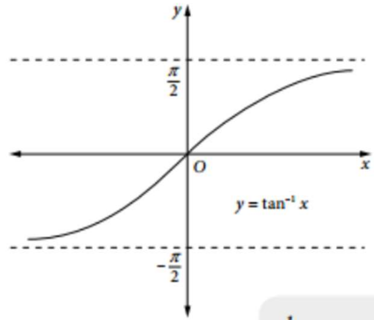


Derivative of $\tan^{-1}x$

$\tan^{-1}x$ is defined for all x.

Let $y=\tan^{-1}x \quad \therefore \quad x=\tan y, -\frac{\pi}{2}<y<\frac{\pi}{2}$

$$\frac{dx}{dy}=\sec^2 y$$
$$=1+\tan^2 y$$
$$\therefore \frac{dy}{dx}=\frac{1}{1+\tan^2 y} \quad \text{for } -\frac{\pi}{2}<y<\frac{\pi}{2}$$
$$=\frac{1}{1+x^2} \quad \text{for all } x$$



$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2} \text{ for all } x$$

Derivative of $\tan^{-1}\frac{x}{a}$

$\tan^{-1}\frac{x}{a}$ is defined for all x.

Let $y=\tan^{-1}\frac{x}{a} \quad \therefore \quad x=a\tan y, -\frac{\pi}{2}<y<\frac{\pi}{2}$

$$\frac{dx}{dy}=a\sec^2 y$$
$$=a(1+\tan^2 y)$$
$$\therefore \frac{dy}{dx}=\frac{1}{a(1+\tan^2 y)} \quad \text{for } -\frac{\pi}{2}<y<\frac{\pi}{2}$$
$$=\frac{a}{a^2+x^2} \quad \text{for all } x$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right)=\frac{a}{a^2+x^2} \text{ for all } x$$

Example 22

Find the derivative of $x\cos^{-1}(2x+1)$, stating the values of x for which the derivative is defined.

Solution

Following the previous example and using the product rule:

$$\frac{d}{dx}\left(x\cos^{-1}(2x+1)\right)=x\times\frac{-2}{\sqrt{-4x(x+1)}}+\cos^{-1}(2x+1)$$
$$=\frac{-2x}{\sqrt{-4x(x+1)}}+\cos^{-1}(2x+1)$$

On the RHS, the first term is defined for $-1<x<0$ and the second term is defined for $-1\le x\le 0$. Thus the complete RHS is defined for $-1<x<0$ and these are the values for x for which the derivative is defined.

Example 23

Differentiate $\sin^{-1}(\cos x)$. Hence sketch the graph of $y = \sin^{-1}(\cos x)$ for $-\pi \le x \le \pi$.

Solution

Let $y = \sin^{-1}(\cos x)$

$= \sin^{-1} u \quad$ where $u = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times (-\sin x)$$

$$= \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}}$$

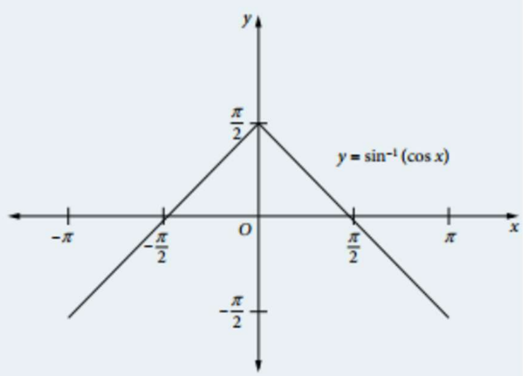
$$= \frac{-\sin x}{|\sin x|}$$

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
y	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$

Now $|\sin x| = \sin x \quad$ for $\sin x \ge 0$, i.e. for $0 \le x \le \pi$

$= -\sin x \quad$ for $\sin x \le 0$, i.e. for $-\pi \le x \le 0$

$$\therefore \frac{dy}{dx} = \begin{cases} \frac{-\sin x}{\sin x} = -1 & \text{for } 0 < x < \pi \\ \frac{-\sin x}{-\sin x} = 1 & \text{for } -\pi < x < 0 \end{cases}$$



$\frac{dy}{dx}$ is not defined when $x = -\pi, 0, \pi$. If there were no restrictions on the domain, the graph would repeat itself (i.e. it would be periodic with period 2π). The range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and the derivative is not defined for any values of $x = n\pi$, $n = 0, \pm1, \pm2, \ldots$

Note that these sharp peaks are *not* turning points, because the function here changes sharply instead of smoothly. The point where the function changes sharply is called a **cusp**.

Example 24

(a) Differentiate $\sin^{-1} x + \cos^{-1} x$. **(b)** Hence show that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Solution

(a) $$\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

(b) $\sin^{-1} x + \cos^{-1} x$ is a constant as its derivative is 0.

The value of the constant can be found by evaluating the function at any x in its domain.

Where $x = 0$: $\quad \sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$

$\therefore \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (You may wish to verify this by substituting other values for x.)