INEQUALITIES

1 Show that if $a \geq 0$ and $b \geq 0$ then $ab(a^2+b^2) \geq 2a^2b^2$.

$$ab(a^2+b^2) \geqslant 2a^2b^2 \iff ab(a^2+b^2) - 2a^2b^2 \geqslant 0$$

$$\iff ab\left[a^2+b^2-2ab\right] \geqslant 0$$

$$\iff ab\left[a-b\right]^2 \geqslant 0$$

which is true, as both the LHS and the RHS of the expression to the left of the inequality are positive.

$\therefore$ indeed $ab(a^2+b^2) \geqslant 2a^2b^2$

2 If $0 < x < y$, prove that $x^2 < xy < y^2$.

$x < y \Rightarrow xy < y^2$ by multiplying both sides by y which is a positive number.

Also $x < y \Rightarrow x^2 < yx$ by multiplying both sides by x which is a positive number.

$\therefore$ $xy < y^2$ and $x^2 < yx$

$\therefore$ $x^2 < xy < y^2$

INEQUALITIES

3 **(a)** For positive x and y, prove that $\frac{x}{y}+\frac{y}{x}\geq 2$. **(b)** Hence prove that $x^2-xy+y^2\geq xy$.

(c) Factorise x^3+y^3 and show that $x^3+y^3\geq xyz\left(\frac{x}{z}+\frac{y}{z}\right)$ for $x, y, z>0$.

(d) Write similar expressions for y^3+z^3 and z^3+x^3.

(e) Using results from parts **(c)** and **(d)**, prove that $x^3+y^3+z^3\geq 3xyz$.

(f) If a, b, c, d are positive, deduce that:

(i) $a+b+c\geq 3\sqrt[3]{abc}$ **(ii)** $(a+b+c)(a+b+d)(a+c+d)(b+c+d)\geq 81abcd$

a) $\frac{x}{y}+\frac{y}{x}\geqslant 2 \iff \frac{x^2}{y}+y\geqslant 2x \iff x^2+y^2\geqslant 2xy$ ①

$\iff (x-y)^2\geqslant 0$ which is true.

b) From ① $x^2+y^2\geqslant 2xy \iff x^2-xy+y^2\geqslant xy$.

c) $x^3+y^3=(x+y)(x^2-xy+y^2)$

As x and $y>0$ $\quad x^2-xy+y^2\geqslant xy \iff (x+y)(x^2-xy+y^2)\geqslant xy(x+y)$

$\iff x^3+y^3\geqslant xy(x+y)$

$\iff x^3+y^3\geqslant xyz\left(\frac{x}{z}+\frac{y}{z}\right)$ Ⓐ

d) Likewise: $y^3+z^3\geqslant xyz\left(\frac{y}{x}+\frac{z}{x}\right)$ Ⓑ and $z^3+x^3\geqslant xyz\left(\frac{z}{y}+\frac{x}{y}\right)$ Ⓒ

e) Adding Ⓐ, Ⓑ and Ⓒ, we obtain

$2(x^3+y^3+z^3)\geqslant xyz\left[\frac{x}{z}+\frac{y}{z}+\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}\right]$

But, we demonstrated at a) that $\frac{x}{y}+\frac{y}{x}\geqslant 2$, so:

$xyz\left(\frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{y}+\frac{y}{x}\right)\geqslant xyz(2+2+2)$

$\therefore x^3+y^3+z^3\geqslant xyz\times\frac{6}{2}$ or $x^3+y^3+z^3\geqslant 3xyz$

f) i) let $a=x^3$ $b=y^3$ and $c=z^3$ so $a+b+c\geqslant 3\sqrt[3]{a}\sqrt[3]{b}\sqrt[3]{c}$

or $a+b+c\geqslant 3\sqrt[3]{abc}$

ii) Likewise: $a+b+d\geqslant 3\sqrt[3]{abd}$

$a+c+d\geqslant 3\sqrt[3]{acd}$ By multiplying the 4 inequalities,

and $b+c+d\geqslant 3\sqrt[3]{bcd}$ we obtain

$(a+b+c)(a+b+d)(a+c+d)(b+c+d)\geqslant 81\,abcd$

4 **(a)** Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for real a, b, c.
(b) Hence show that $(a + b + c)^2 \geq 3(ab + bc + ca)$.

a) $a^2 + b^2 + c^2 \geqslant ab + bc + ca$

We know $(a-b)^2 \geqslant 0 \Leftrightarrow a^2 - 2ab + b^2 \geqslant 0$

$(b-c)^2 \geqslant 0 \Leftrightarrow b^2 - 2bc + c^2 \geqslant 0$

$(a-c)^2 \geqslant 0 \Leftrightarrow a^2 - 2ac + c^2 \geqslant 0$

$\therefore$ adding the 3 inequalities: $2(a^2+b^2+c^2) - 2(ab+bc+ac) \geqslant 0$

or $a^2 + b^2 + c^2 \geqslant ab + bc + ac$

b) $(a+b+c)^2 = (a+b+c)(a+b+c)$

$= a^2 + ab + ac + ba + b^2 + bc + ac + bc + c^2$

$= a^2 + b^2 + c^2 + 2(ab + ac + bc).$

But $a^2 + b^2 + c^2 \geqslant ab + bc + ac$

as demonstrated at a).

$\therefore$ $a^2 + b^2 + c^2 + 2(ab + ac + bc) \geqslant 3(ab + bc + ac)$

$\therefore$ $(a + b + c)^2 \geqslant 3(ab + bc + ac)$

6 **(a)** Prove that $\frac{a+b}{2} \geq \sqrt{ab}$. Hence prove that $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ for positive a, b, c, d.

(b) Let $d = \frac{a+b+c}{3}$. Show that $abc \leq \left(\frac{a+b+c}{3}\right)^3$.

a) $(\sqrt{a} - \sqrt{b})^2 \geqslant 0 \iff a - 2\sqrt{ab} + b \geqslant 0$

$\iff a + b \geqslant 2\sqrt{ab}$

$\iff \frac{a+b}{2} \geqslant \sqrt{ab}$ Ⓐ

Similarly $\frac{c+d}{2} \geqslant \sqrt{cd}$ Ⓑ

Adding Ⓐ and Ⓑ we obtain $\frac{a+b+c+d}{2} \geqslant \sqrt{ab} + \sqrt{cd}$

or $\frac{a+b+c+d}{4} \geqslant \frac{\sqrt{ab} + \sqrt{cd}}{2} \geqslant \sqrt{\sqrt{ab}\sqrt{cd}}$.

So $\frac{a+b+c+d}{4} \geqslant \sqrt[4]{abcd}$

b) The inequality becomes: $\frac{a+b+c+\frac{1}{3}(a+b+c)}{4} \geqslant \sqrt[4]{abc\left(\frac{1}{3}(a+b+c)\right)}$

or $\frac{4}{3 \times 4}(a+b+c) \geqslant \sqrt[4]{abc}\sqrt[4]{\frac{a+b+c}{3}}$

or $\frac{a+b+c}{3} \geqslant \sqrt[4]{abc}\sqrt[4]{\frac{a+b+c}{3}}$

or $\left(\frac{a+b+c}{3}\right)^4 \geqslant abc \times \left(\frac{a+b+c}{3}\right)$

or $\left(\frac{a+b+c}{3}\right)^3 \geqslant abc$

8 If $x > 0$ and $y > 0$, prove that: **(a)** $\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}$ **(b)** $\frac{1}{x^2}+\frac{1}{y^2}\geq\frac{8}{(x+y)^2}$

a) $\frac{1}{x}+\frac{1}{y} \geqslant \frac{4}{x+y} \iff \frac{x+y}{xy} \geqslant \frac{4}{x+y}$

$\iff (x+y)^2 \geqslant 4xy$

$\iff x^2+2xy+y^2 \geqslant 4xy$

$\iff x^2-2xy+y^2 \geqslant 0$

$\iff (x-y)^2 \geqslant 0$ which is true.

So indeed we must have $\frac{1}{x}+\frac{1}{y} \geqslant \frac{4}{x+y}$.

b) Squaring both sides: $\frac{1}{x^2}+\frac{1}{y^2}+\frac{2}{xy} \geqslant \frac{16}{(x+y)^2}$

But also $\left(\frac{1}{x}-\frac{1}{y}\right)^2 \geqslant 0 \iff \frac{1}{x^2}+\frac{1}{y^2}-\frac{2}{xy} \geqslant 0$

Adding both: $2\left[\frac{1}{x^2}+\frac{1}{y^2}\right] \geqslant \frac{16}{(x+y)^2}$

$\frac{1}{x^2}+\frac{1}{y^2} \geqslant \frac{8}{(x+y)^2}$

9 **(a)** If a and b are real numbers, prove that $4a^2 - 6ab + 4b^2 \geq a^2 + b^2$.
(b) Write the binomial expansion of $(a-b)^4$ and prove that $a^4 + b^4 \geq a^3b + ab^3$ if $a > 0$ and $b > 0$.

a) $4a^2 - 6ab + 4b^2 \geqslant a^2 + b^2$ (A)

$\Leftrightarrow 3a^2 - 6ab + 3b^2 \geqslant 0$

$\Leftrightarrow a^2 - 2ab + b^2 \geqslant 0$

$\Leftrightarrow (a-b)^2 \geqslant 0$ which is true, Therefore the inequality (A) must also be true.

1 2 1
1 3 3 1
1 4 6 4 1

b) $(a-b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$= a^4 + b^4 - ab(4a^2 - 6ab + 4b^2)$

$(a-b)^4 \geqslant 0$ or $a^4 + b^4 - ab(4a^2 - 6ab + 4b^2) \geqslant 0$

$\Leftrightarrow a^4 + b^4 \geqslant ab(4a^2 - 6ab + 4b^2)$

$\Leftrightarrow a^4 + b^4 \geqslant ab(a^2 + b^2)$

So $a^4 + b^4 \geqslant a^3b + ab^3$

14 The area of a triangle is given by Heron's formula as $A = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b and c are the lengths of the sides and $s = \frac{1}{2}(a+b+c)$. Given that $\sqrt{ab} \le \frac{a+b}{2}$ and $ab + bc + ca \le a^2 + b^2 + c^2$, show that: $A \le \frac{a^2+b^2+c^2}{6}$.

$$A \leqslant \frac{s(s-a) + (s-b)(s-c)}{2}$$

$$A \leqslant \frac{s^2 - as + s^2 - bs - sc + bc}{2}$$

$$A \leqslant \frac{2s^2 - s(a+b+c) + bc}{2}$$

$$A \leqslant \frac{2s^2 - s \times 2s + bc}{2} \qquad \text{as } s = \frac{1}{2}(a+b+c)$$

$$\therefore A \leqslant \frac{bc}{2}$$

Likewise $A \leqslant \frac{ab}{2}$ and $A \leqslant \frac{ac}{2}$

So adding the 3 inequalities, we obtain:

$$3A \leqslant \frac{ab + bc + ac}{2}$$

But $ab + bc + ac \leqslant a^2 + b^2 + c^2$, Therefore:

$$3A \leqslant \frac{a^2 + b^2 + c^2}{2}$$

or $$A \leqslant \frac{a^2 + b^2 + c^2}{6}$$

17 Let $g(x) = \sin x - x$.

(a) Show that $g(0) = 0$ and $g'(0) = 0$.

(b) Show that $-2 \leq g'(x) \leq 0$ for all x.

(c) Hence explain why $g(x) \leq 0$ for $x \geq 0$.

(d) Explain why $\sin x < x$ for $x > 0$.

a) $g(0) = \sin 0 - 0 = 0$ $\quad g'(x) = \cos x - 1$ so $g'(0) = 1 - 1 = 0$

b) $-1 \leqslant \cos x \leqslant 1 \quad \forall x$, so

$-2 \leqslant \cos x - 1 \leqslant 0$ or $-2 \leqslant g'(x) \leqslant 0 \quad \forall x$.

c) as $g'(x) \leqslant 0$, that means $g(x)$ is decreasing.

But $g(0) = 0$. So $\forall x > 0$ $g(x)$ must be below 0.

i.e. $g(x) \leqslant 0$ for $x \geqslant 0$

d) $g(x) \leqslant 0$ means $\sin x - x \leqslant 0$. $\forall x \geqslant 0$

or $\sin x \leqslant x \quad \forall x \in \mathbb{R}^+$

or $\sin x < x \quad \forall x > 0$.

19 By letting $a = \frac{1}{x}$ and $b = \frac{1}{y}$ in $\frac{a+b}{2} \geq \sqrt{ab}$, prove that:

(a) $\frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}}$ **(b)** $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{2}{xy}$

a) $\frac{a+b}{2} = \frac{\frac{1}{x} + \frac{1}{y}}{2} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right).$

So $\frac{a+b}{2} \geqslant \sqrt{ab} \iff \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right) \geqslant \sqrt{\frac{1}{x}\frac{1}{y}}$

or $\left(\frac{1}{x} + \frac{1}{y}\right) \geqslant \frac{2}{\sqrt{xy}}$

b) Taking the square on both sides:

$\left(\frac{1}{x} + \frac{1}{y}\right)^2 \geqslant \frac{4}{xy}$ or $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} \geqslant \frac{4}{xy}$

or $\frac{1}{x^2} + \frac{1}{y^2} \geqslant \frac{2}{xy}$

20 If $1 \le x \le 4$, show that: $\frac{1}{3} \le \frac{1}{1+\sqrt{x}} \le \frac{1}{2}$

$$1 \leqslant x \leqslant 4 \iff 1 \leqslant \sqrt{x} \leqslant 2$$

$$\iff 2 \leqslant 1+\sqrt{x} \leqslant 3$$

$$\iff \frac{1}{3} \leqslant \frac{1}{1+\sqrt{x}} \leqslant \frac{1}{2}$$

that was an easy one ... ☺