EXPONENTIALS AND LOGARITHMS - CHAPTER REVIEW (BIS)

1. Find the value of y.

(1) $\log_5 25 = y$ (2) $\log_3 1 = y$ (3) $\log_{16} 4 = y$ (4) $\log_2 \frac{1}{8} = y$

$y = 2$ $y = 0$ $y = 1/2$ $y = -3$

(5) $\log_5 1 = y$ (6) $\log_2 8 = y$ (7) $\log_7 \frac{1}{7} = y$ (8) $\log_3 \frac{1}{9} = y$

$y = 0$ $y = 3$ $y = -1$ $y = -2$

(9) $\log_y 32 = 5$ (10) $\log_9 y = -\frac{1}{2}$ (11) $\log_4 \frac{1}{8} = y$ (12) $\log_9 \frac{1}{81} = y$

$y = 2$ $y = 1/3$ $4^y = 8$ so $2^{2y} = 2^3$ $y = 1.5$ $y = -2$

2. Evaluate.

(1) $\log_3 1$ (2) $\log_4 4$ (3) $\log_7 7^3$ (4) $b^{\log_b 3}$ (3) $\log_{25} 5^3 = x$ (4) $16^{\log_4 8} = x$

0 1 3 3

$25^x = 5^3$
$5^{2x} = 5^3$ $x = 1.5$

$x = 16^{\log_4 2^3}$
$x = 16^{3\log_4 2}$
$x = 16^{3 \times 1/2}$
$x = 16^{1.5} = 64$

3. Write the following expressions in terms of logs of x, y and z.

(1) $\log x^2 y$ (2) $\log \frac{x^3 y^2}{z}$ (3) $\log \frac{\sqrt{x}\sqrt[3]{y^2}}{z^4}$ (4) $\log xyz$

$= \log x^2 + \log y$
$= 2\log x + \log y$

$= 3\log x + 2\log y - \log z$

$= \frac{1}{2}\log x + \frac{2}{3}\log y - 4\log z$

$= \log x + \log y + \log z$

(5) $\log \frac{x}{yz}$ (6) $\log \left(\frac{x}{y}\right)^2$ (7) $\log (xy)^{\frac{1}{3}}$ (8) $\log x\sqrt{z}$

(5) $= \log x - \log y - \log z$

(6) $= 2[\log x - \log y]$

(7) $= \frac{1}{3}[\log x + \log y]$

(8) $= \log x + \frac{1}{2}\log z$

(9) $\log \dfrac{\sqrt[3]{x}}{\sqrt[3]{yz}}$ (10) $\log \sqrt[4]{\dfrac{x^3y^2}{z^4}}$ (11) $\log x\sqrt{\dfrac{\sqrt{x}}{z}}$ (12) $\log \sqrt{\dfrac{xy^2}{z^8}}$

(9) $= \frac{1}{3}\log x - \frac{1}{3}\left[\log y + \log z\right] = \frac{1}{3}\left[\log x - \log y - \log z\right]$

(10) $= \frac{1}{4}\left[3\log x + 2\log y - 4\log z\right]$

(11) $= \log x + \frac{1}{2}\log\frac{\sqrt{x}}{z} = \log x + \frac{1}{2}\left[\frac{1}{2}\log x - \log z\right] = \frac{5}{4}\log x - \frac{1}{2}\log z$

(12) $= \frac{1}{2}\left[\log x + 2\log y - 8\log z\right] = \frac{1}{2}\log x + \log y - 4\log z$

4. Write the following equalities in exponential form.

(1) $\log_3 81 = 4$ (2) $\log_7 7 = 1$ (3) $\log_{\frac{1}{2}} \frac{1}{8} = 3$ (4) $\log_3 1 = 0$

$3^4 = 81$ $7^1 = 7$ $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ $3^0 = 1$

(5) $\log_4 \frac{1}{64} = -3$ (6) $\log_6 \frac{1}{36} = -2$ (7) $\log_x y = z$ (8) $\log_m n = \frac{1}{2}$

$4^{-3} = \frac{1}{64}$ $6^{-2} = \frac{1}{36}$ $x^z = y$ $m^{1/2} = n$

5. Write the following equalities in logarithmic form.

(1) $8^2 = 64$ (2) $10^3 = 1000$ (3) $4^{-2} = \frac{1}{16}$ (4) $3^{-4} = \frac{1}{81}$

$\log_8 64 = 2$ $\log_{10} 10{,}000 = 3$ $\log_4\left(\frac{1}{16}\right) = -2$ $\log_3 \frac{1}{81} = -4$

(5) $\left(\frac{1}{2}\right)^{-5} = 32$ (6) $\left(\frac{1}{3}\right)^{-3} = 27$ (7) $x^{2z} = y$ (8) $\sqrt{x} = y$

$\log_{\frac{1}{2}} 32 = -5$ $\log_{1/3} 27 = -3$ $\log_x y = 2z$ $\log_x y = 1/2$

6. True or False?

(1) $\log\left(\dfrac{x}{y^3}\right) = \log x - 3\log y$ (2) $\log(a-b) = \log a - \log b$ (3) $\log x^k = k \cdot \log x$

(1) $\log\left(\dfrac{x}{y^3}\right) = \log x - \log y^3 = \log x - 3\log y$ so true

(2) $\log(a-b) \neq \log a - \log b$ so false

(3) $\log x^k = k \log x$ so TRUE

(4) $(\log a)(\log b) = \log(a+b)$ (5) $\dfrac{\log a}{\log b} = \log(a-b)$ (6) $(\ln a)^k = k \cdot \ln a$

FALSE FALSE FALSE

(7) $\log_a a^a = a$ (8) $-\ln\left(\frac{1}{x}\right) = \ln x$

(7) $\log_a a^a = a \times \underbrace{\log_a a}_{=1} = a \times 1 = a$ so TRUE

(8) $-\ln\left(\frac{1}{x}\right) = -\ln x^{-1} = -(-1)\ln x = \ln x$ so TRUE

7. Solve the following logarithmic equations.

(1) $\ln x = -3$ (2) $log_2(3x - 2) = 2$

(1) $\ln x = \log_e x$ so $x = e^{-3} = 1/e^3$

(2) $(3x-2) = 2^2 = 4$

$\iff 3x - 2 = 4 \iff 3x = 6$ so $x = 2$

(3) $2\log x = \log 2 + \log(3x - 4)$ (4) $\log x + \log(x-1) = \log(4x)$

(3) $\iff \log x^2 = \log 2(3x-4)$ so $x^2 = 2(3x-4)$

$\iff x^2 - 6x + 8 = 0$ $\Delta = (-6)^2 - 4 \times 8 = 36 - 32 = 4 = 2^2$

so $x_1 = \frac{6-2}{2} = \frac{4}{2} = 2$ OR $x_2 = \frac{6+2}{2} = \frac{8}{2} = 4$

The two solutions are possible.

(4) $x(x-1) = 4x \iff x^2 - x = 4x \iff x^2 - 5x = 0$

$\iff x(x-5) = 0$ so $x = 0$ or $x = 5$

Note that $x = 0$ is <u>not</u> possible as the original equation includes $\log x$ and we cannot calculate $\log 0$.

(5) $\log_3(x+25) - \log_3(x-1) = 3$ (6) $\log_9(x-5) + \log_9(x+3) = 1$

(5) $\Leftrightarrow \log_3\left(\frac{x+25}{x-1}\right) = 3$ so $\frac{x+25}{x-1} = 3^3 = 27$

so $x + 25 = 27(x-1) \Leftrightarrow x + 25 = 27x - 27 \Leftrightarrow 26x = 52$

so $x = 52/26 = 2$

(6) $\Leftrightarrow \log_9 (x-5)(x+3) = 1 = \log_9 9$

so we must have $(x-5)(x+3) = 9$

$\Leftrightarrow x^2 - 5x + 3x - 15 = 9 \Leftrightarrow x^2 - 2x - 24 = 0$

$\Delta = (-2)^2 - 4 \times (-24) = 100 = 10^2$ so

either $x_1 = \frac{-(-2)-10}{2} = \frac{-8}{2} = -4$ (impossible as we can't calculate $\log_9(-1)$) OR $x_2 = \frac{2+10}{2} = 6$

(7) $log_{10}x + log_{10}(x-3) = 1$ (8) $\log_2(x-2) + \log_2(x+1) = 2$

(7) $\log_{10}[x(x-3)] = \log_{10} 10$ so $x(x-3) = 10 \Leftrightarrow x^2 - 3x - 10 = 0$

$\Delta = (-3)^2 - 4 \times (-10) = 49 = 7^2$

$x_1 = \frac{3-7}{2} = \frac{-4}{2} = -2$ [impossible as we cannot calculate $\log_{10}(-2)$]

or $x_2 = \frac{3+7}{2} = \frac{10}{2} = 5$ (only possible solution)

(8) $\Leftrightarrow \log_2 (x-2)(x+1) = 2 = \log_2 4$

so $(x-2)(x+1) = 4 \Leftrightarrow x^2 - 2x + x - 2 = 4$

$\Leftrightarrow x^2 - x - 6 = 0$ $\Delta = (-1)^2 - 4 \times (-6) = 1 + 24 = 25 = 5^2$

so $x_1 = \frac{1+5}{2} = \frac{6}{2} = 3$

OR $x_2 = \frac{1-5}{2} = -\frac{4}{2} = -2$ (which is impossible as we cannot calculate $\log_2(-4)$ in the original equation)

8. Prove the following statements.

(1) $\log_{\sqrt{b}} x = 2\log_b x$ (2) $\log_{\frac{1}{\sqrt{b}}} \sqrt{x} = -\log_b x$ (3) $\log_{b^4} x^2 = \log_b \sqrt{x}$

(1) $\log_{\sqrt{b}} x = \dfrac{\log_b x}{\log_b \sqrt{b}}$ (using the change of base rule)

— $= \dfrac{\log_b x}{\frac{1}{2}\log_b b} = \dfrac{\log_b x}{\frac{1}{2}} = 2\log_b x$

(2) $\log_{1/\sqrt{b}} \sqrt{x} = \dfrac{\log_b \sqrt{x}}{\log_b 1/\sqrt{b}}$ (using the change of base rule)

— $= \dfrac{\log_b x^{1/2}}{\log_b b^{-1/2}} = \dfrac{\frac{1}{2}\log_b x}{-\frac{1}{2}\log_b b} = \dfrac{\log_b x}{-\log_b b} = \dfrac{\log_b x}{-1}$

$= -\log_b x$

(3) $\log_{b^4} x^2 = \dfrac{\log_b x^2}{\log_b b^4}$ (using the change of base rule)

— $= \dfrac{2\log_b x}{4 \times \log_b b} = \dfrac{\log_b x}{2 \times \log_b b}$

— $= \dfrac{\log_b x}{2} = \dfrac{1}{2} \times \log_b x = \log_b x^{1/2}$

So indeed $\log_{b^4} x^2 = \log_b \sqrt{x}$

EXPONENTIALS AND LOGARITHMS - CHAPTER REVIEW II

9. Given that $\log 2 = x$, $\log 3 = y$ and $\log 7 = z$, express the following expressions in terms of x, y, and z.

(1) $log\ 12$ (2) $log\ 392$ (3) $log\left(\frac{14}{3}\right)$ (4) $log\left(\frac{6}{7}\right)$

(1) $\log 12 = \log(2^2 \times 3) = \log 2^2 + \log 3 = 2\log 2 + \log 3 = 2x + y$

(2) $\log 392 = \log(2^3 \times 7^2) = 3\log 2 + 2\log 7 = 3x + 2z$

(3) $\log\left(\frac{14}{3}\right) = \log 14 - \log 3 = \log(2\times 7) - y = \log 2 + \log 7 - y = x + z - y$

(4) $\log\left(\frac{6}{7}\right) = \log(2\times 3) - \log 7 = \log 2 + \log 3 - \log 7$

$= x + y - z$

(5) $log\ 1.5$ (6) $log\ 10.5$ (7) $log\ 24.5$ (8) $log\left(\frac{7776}{7}\right)$

(5) $\log 1.5 = \log\left(\frac{3}{2}\right) = \log 3 - \log 2 = y - x$

(6) $\log 10.5 = \log \frac{21}{2} = \log\left(\frac{3\times 7}{2}\right) = \log 3 + \log 7 - \log 2 = y + z - x$

(7) $\log 24.5 = \log \frac{49}{2} = \log \frac{7^2}{2} = 2\log 7 - \log 2 = 2z - x$

(8) $\log\left(\frac{7776}{7}\right) = \log\left(\frac{2^5 \times 3^5}{7}\right) = \log 2^5 + \log 3^5 - \log 7$

$= 5\log 2 + 3\log 3 - \log 7$

$= 5x + 3y - z$

10. Solve the following equations.

(1) $3^x - 2 = 12$

$3^x = 14$

$x = \frac{\ln 14}{\ln 3}$

(2) $3^{1-x} = 2$

$\ln 3^{1-x} = \ln 2$

$(1-x)\ln 3 = \ln 2$

$1 - x = \frac{\ln 2}{\ln 3}$ so $x = 1 - \frac{\ln 2}{\ln 3}$

(3) $4^x = 5^{x+1}$

$\ln 4^x = \ln 5^{x+1}$

$x \ln 4 = (x+1)\ln 5$

$x[\ln 4 - \ln 5] = \ln 5$

$x \ln(4/5) = \ln 5$

$x = \frac{\ln 5}{\ln(4/5)}$

(4) $6^{1-x} = 10^x$

$\ln 6^{1-x} = \ln 10^x$

$(1-x)\ln 6 = x \ln 10$

$x[\ln 10 + \ln 6] = \ln 6$

$x = \frac{\ln 6}{\ln 60}$

(5) $3^{2x+1} = 2^{x-2}$

$\ln 3^{2x+1} = \ln 2^{x-2}$

$(2x+1)\ln 3 = (x-2)\ln 2$

$x[\ln 3^2 - \ln 2] = -2\ln 2 - \ln 3$

$x \times \ln\left(\frac{9}{2}\right) = \ln 2^{-2} - \ln 3$

$x \times \ln\left(\frac{9}{2}\right) = \ln\left(\frac{1}{4\times 3}\right)$

$x = \frac{-\ln 12}{\ln(9/2)}$

(6) $\frac{10}{1+e^{-x}} = 2$

$1 + e^{-x} = 5$

$e^{-x} = 4$

$\ln e^{-x} = \ln 4$

$-x = \ln 4$

so $x = -\ln 4 = \ln \frac{1}{4}$

13. 15 000$ is invested in an account that yeilds 5% interest per year. After how many years will the account be worth 91 221.04$ if the interest is compounded yearly?

$$91{,}221.04 = 15{,}000\left(1+0.05\right)^n$$

So $1.05^n = \dfrac{91{,}221.04}{15{,}000}$

$$\ln\left(1.05^n\right) = \ln\left(\frac{91{,}221.04}{15{,}000}\right) \simeq 1.805$$

so $n \times \ln 1.05 \simeq 1.805$

$$n = \frac{1.805}{\ln 1.05} \simeq 37 \text{ years.}$$

14. 8 000$ is invested in an account that yeilds 6% interest per year. After how many years will the account be worth 13709.60$ if the interest is compounded monthly?

6% per year so it's 0.5% per month.

$$13{,}709.60 = 8000\left(1+0.005\right)^n$$

$$1.005^n = \frac{13{,}709.60}{8000}$$

$$n \times \ln 1.005 = \ln\left[\frac{13709.60}{8000}\right]$$

$$n = \frac{\ln\left[\frac{13709.60}{8000}\right]}{\ln\left[1.005\right]} = 108 \text{ months}$$

which is 9 years.

15. Starting at the age of 40, an average man loses 5% of his hair every year. At what age should an average man expect to have half his hair left?

If at 40, the number of hair is N_0
we want to find when the number of hair is $N_0/2$
$N = N_0 \times 0.95^n$ where n is the number of years.
For N to be equal to $N_0/2$, we must have
$\frac{N_0}{2} = N_0 \times 0.95^n$ so $0.95^n = 0.5$
so $n \times \ln 0.95 = \ln 0.5$
$n = \ln 0.5 / \ln 0.95 = 13.5$ years

16. A bacteria culture starts with 10 00 bacteria and the number doubles every 40 minutes.

(a) Find a formula for the number of bacteria at time t.
(b) Find the number of bacteria after one hour.
(c) After how many minutes will there be 50 000 bacteria?

a) $N = 1000 \times 2^n$ where n is the multiple of 40 minutes.
b) when $t = 1$ hour, i.e $\frac{60}{40} = 1.5$ period,
the number of bacteria is $N = 1{,}000 \times 2^{1.5} = 2828$
c) $50{,}000 = 1000 \times 2^n$ so $2^n = 50$
$n \cdot \ln 2 = \ln 50$ so $n = \frac{\ln 50}{\ln 2} = 5.6438$
so in $5.6438 \times 40 = 226$ minutes approx