OTHER INDUCTION QUESTIONS

There is quite a range of situations in which you will use mathematical induction to prove results.

Example 20

Prove by induction that $3^n > 1 + 2n$ for all integers $n > 1$ (i.e. prove that $3^n - 1 - 2n > 0$).

Solution

Let $S(n)$ be the statement that $3^n - 1 - 2n > 0$ for integer n.

Step 1 Prove that $S(2)$ is true. (Note that $n = 2$ is the first case.)

$$\begin{aligned} \text{LHS} &= 3^2 - 1 - 4 \\ &= 4 \\ &> 0 \qquad \therefore S(2) \text{ is true.} \end{aligned}$$

Step 2 Assume $S(k)$ is true for an integer $k \geq 2$.

i.e. assume that $\quad 3^k - 1 - 2k > 0 \qquad$ [a]

Now prove that $S(k + 1)$ is true if $S(k)$ is true.

i.e. prove that $\quad 3^{k+1} - 1 - 2(k + 1) > 0$

$$\begin{aligned} \text{LHS} &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3 \times 3^k - 2k - 3 \end{aligned}$$

We need to link [a] to this, so we need to group $3^k - 1 - 2k$ together. However, the term in 3^k is multiplied by 3, so we need $3(3^k - 2k - 1)$. Form this group and 'pay back' the extra terms as required:

$$\begin{aligned} \text{LHS} &= 3 \times 3^k - 6k - 3 + 4k \\ &= 3(3^k - 2k - 1) + 4k \\ &> 0 \qquad \text{as } 3^k - 1 - 2k > 0 \text{ from [a] and } k > 0 \end{aligned}$$

Step 3 Conclusion

$S(k + 1)$ is true if $S(k)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

Example 22

Prove that $3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n.

Solution

Step 1 $n = 1$: LHS $= 3^6 - 2^2 = 729 - 4 = 725$, which is divisible by 5.

Hence the result is true when $n = 1$.

Step 2 Assume the result is true for $n = k$, i.e. assume that $3^{2k+4} - 2^{2k} = 5M$, where M is a positive integer.

Prove the result is true for $n = k + 1$, i.e. prove that $3^{2k+6} - 2^{2k+2}$ is divisible by 5.

$$\begin{aligned} \text{Exp} &= 3^{2k+6} - 2^{2k+2} \\ &= 9 \times 3^{2k+4} - 4 \times 2^{2k} \\ &= 9 \times 3^{2k+4} - 9 \times 2^{2k} + 5 \times 2^{2k} \\ &= 9 \times 5M + 5 \times 2^{2k} \\ &= 5(9M + 2^{2k}), \text{ which is divisible by 5.} \end{aligned}$$

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

Example 21

Prove that $n^2 \geq 2n + 1$ for positive integers $n \geq 3$.

Solution

Let $S(n)$ be the statement that $n^2 - (2n + 1) > 0$.

Step 1 Prove that $S(3)$ is true.

$$\begin{aligned} \text{LHS} &= 3^2 - 6 - 1 \\ &= 9 - 7 \\ &= 2 > 0 \end{aligned}$$

Hence $S(3)$ is true.

Step 2 Assume that $S(k)$ is true, i.e. assume that $k^2 - 2k - 1 > 0$.

Prove that $S(k + 1)$ is true if $S(k)$ is true, i.e. $(k + 1)^2 - 2(k + 1) - 1 > 0$.

$$\begin{aligned} \text{LHS} &= (k + 1)^2 - 2(k + 1) - 1 \\ &= k^2 + 2k + 1 - 2k - 2 - 1 \\ &= k^2 - 2k - 1 + 2k - 1 \\ &> 0 + 2k - 1 \\ &> 0 \text{ as } 2k - 1 > 5 \text{ when } k \geq 3. \end{aligned}$$

Hence $S(k + 1)$ is true if $S(k)$ is true.

Step 3 But $S(3)$ is true so by the principle of mathematical induction $S(n)$ is true for all $n \geq 3$.

Example 23

Prove that every integer greater than 1 is either prime or a product of primes.

Solution

Let $S(n)$ be the proposition that n is either prime or a product of primes, $n \geq 2$.

Step 1 2 is a prime $\therefore$ $S(2)$ is true.

Step 2 Assume $S(2), S(3), S(4), \ldots S(k)$ are true. Thus prove that $S(k + 1)$ is true.

i.e. assume 3 is either prime or a product of primes
assume 4 is either prime or a product of primes
assume 5 is either prime or a product of primes
…
assume k is either prime or a product of primes.

Now $k + 1$ is either prime, in which case $S(k + 1)$ is true, or $k + 1$ is composite, in which case $k + 1 = p \times q$ where p and q are integers less than k (so that both p and q are either prime or a product of primes, because both p and q are in the set of assumed primes or products of primes.)

$\therefore$ $k + 1 = p \times q$ is a product of primes and $S(k + 1)$ is true.

Step 3 *Conclusion*

$S(k + 1)$ is true if $S(2), S(3), S(4), \ldots S(k)$ are true.
$S(2)$ is true.
$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

Example 24

Construct a proof by induction of the geometrical property that 'the angle sum of an n-sided polygon is $(n-2) \times 180°$ for all integers $n \geq 3$'.

Solution

Let $S(n)$ be the statement that the angle sum of an n-sided polygon is $(n-2) \times 180°$ for integer n.

Step 1 Prove that $S(3)$ is true.

When $n = 3$, angle sum $= (3-2) \times 180° = 180°$, which is the angle sum of a triangle.

$\therefore$ $S(3)$ is true.

Step 2 Assume $S(k)$ is true for an integer $k > 3$.

i.e. assume that the angle sum of a k-sided polygon is $(k-2) \times 180°$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that the angle sum of a $(k+1)$-sided polygon is $([k+1]-2) \times 180°$

$= (k-1) \times 180°$

In the $(k+1)$-sided polygon, construct a diagonal that divides the polygon into a k-sided polygon and a triangle. (This can always be done.)

Angle sum of $(k+1)$-sided polygon = (angle sum of k-sided polygon) + (angle sum of triangle)

using [a]: $= (k-2) \times 180° + 180°$

$= (k-1) \times 180°$ (as required)

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(3)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 3$.