DEFINITE INTEGRALS INVOLVING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1 Find the value of:

(a) $\int_{-1}^{1} e^x\,dx$ (b) $\int_{0}^{2} e^{2x}\,dx$ (c) $\int_{-1}^{3} e^{-\frac{x}{2}}\,dx$ (d) $\int_{0}^{1} e^{1.5t}\,dt$

a) $\int_{-1}^{1} e^x\,dx = \left[e^x\right]_{-1}^{1} = e^1 - e^{-1} = e - 1/e$

b) $\int_0^2 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^2 = \frac{e^4}{2} - \frac{e^0}{2} = \frac{e^4-1}{2}$

c) $\int_{-1}^{3} e^{-x/2}\,dx = \left[\frac{e^{-x/2}}{(-1/2)}\right]_{-1}^{3} = \left[-2e^{-x/2}\right]_{-1}^{3} = -2e^{-3/2} - (-2e^{1/2})$

$= 2\left[\sqrt{e} - \frac{1}{e\sqrt{e}}\right]$

d) $\int_0^1 e^{1.5t}\,dt = \left[\frac{e^{1.5t}}{1.5}\right]_0^1 = \frac{e^{1.5}}{1.5} - \frac{e^0}{1.5}$

$= \frac{e\sqrt{e}-1}{1.5}$

2 Indicate whether each statement below is a correct or incorrect step in the evaluation of $I = \int_{-1}^{1}\left(e^x - e^{-x}\right)^2 dx$.

$\boxed{\text{(a)}}$ $I = \int_{-1}^{1}\left(e^{2x} - 2 + e^{-2x}\right)dx$ $\boxed{\text{(b)}}$ $I = \left[\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2}\right]_{-1}^{1}$

(c) $I = \left[e^{2x} - 4x - e^{-2x}\right]_0^1$ (d) $I = \frac{e^4 - 4e^2 - 1}{4e^2}$

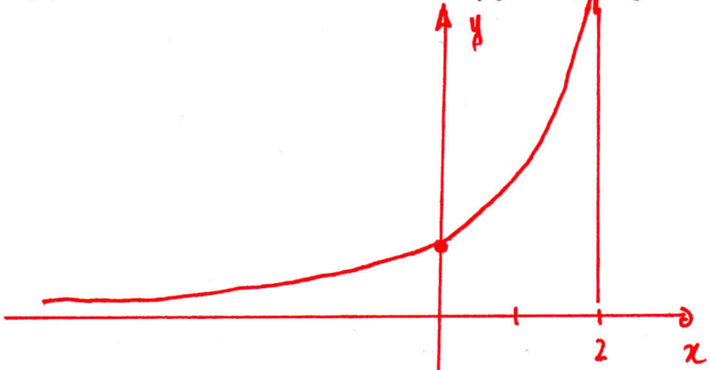
$I = \int_{-1}^{1}(e^x - e^{-x})^2\,dx = \int_{-1}^{1}(e^{2x} - 2 + e^{-2x})\,dx$

$I = \left[\frac{e^{2x}}{2} - 2x + \frac{e^{-2x}}{(-2)}\right]_{-1}^{1} = \left(\frac{e^2 - 4 - e^{-2}}{2}\right) - \left(\frac{e^{-2} + 4 - e^2}{2}\right)$

$I = e^2 - 4 - e^{-2} = \frac{e^4 - 4e^2 - 1}{e^2}$

in fact $\boxed{C}$ is correct too as the function is even.

4 **(a)** Calculate the area bounded by the curve $y = e^x$, the coordinate axes and the line $x = 2$.
(b) Write the equation of the tangent to $y = e^x$ at the point where $x = 2$.
(c) Calculate the area bounded by $y = e^x$, the coordinate axes and the tangent at $x = 2$.
(d) Calculate the area bounded by $y = e^x$, the y-axis and the line $y = e^2$.



a) $\int_0^2 e^x\,dx = \left[e^x\right]_0^2 = e^2 - 1$

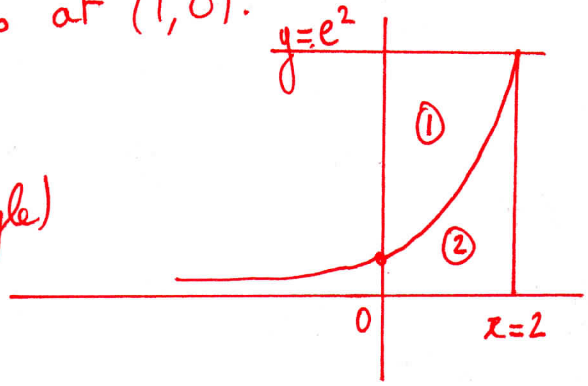
b) the derivative of e^x is e^x. So at $x = 2$, the gradient of the tangent to $y = e^x$ has for gradient e^2. This line passes through $(2, e^2)$ so its equation is $y - e^2 = e^2(x-2)$
or $y = e^2x - e^2$

c) We are looking for $\int_0^2 e^x - (e^2x - e^2)\,dx$

$\int_0^2 \left[e^x - (e^2x - e^2)\right]dx = \left[e^x - \frac{e^2x^2}{2} + e^2x\right]_0^2$

$= e^2 - 2e^2 + 2e^2 - (e^0) = e^2 - 1$

d) the curve $y = e^x$ cuts the y-axis at $(1,0)$.



We want to find Area 1

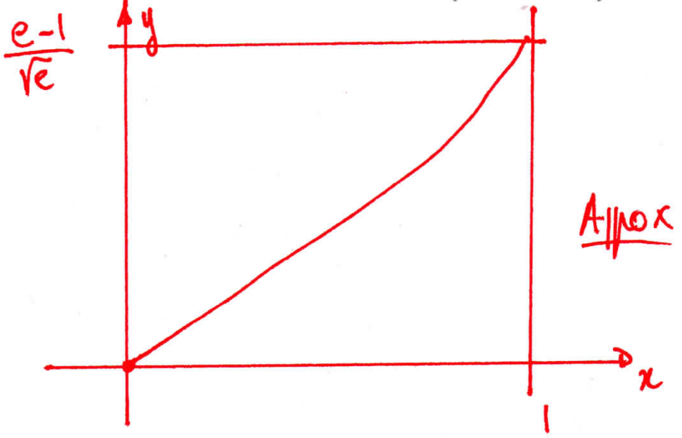
But Area 1 + Area 2 $= 2e^2$ (rectangle)

So Area 1 $= 2e^2 -$ Area 2

$= 2e^2 - (e^2 - 1)$ as calculated at a)

$= e^2 + 1$

6 Calculate the area bounded by the curve $y = e^{0.5x} - e^{-0.5x}$, the x-axis and the line $x = 1$.



For $x = 0$ $\quad y = e^0 - e^0 = 0$

$x = 1$ $\quad y = \sqrt{e} - \frac{1}{\sqrt{e}} = \frac{e-1}{\sqrt{e}} \simeq 1.04$

$$\int_0^1 e^{0.5x} - e^{-0.5x}\, dx = \left[\frac{e^{0.5x}}{0.5} - \frac{e^{-0.5x}}{(-0.5)}\right]_0^1$$

$$= \left(\frac{e^{0.5} + e^{-0.5}}{0.5}\right) - \left(\frac{1+1}{0.5}\right)$$

$$= 2\left[e^{0.5} + e^{-0.5} - 2\right]$$

$$= 2\left[\sqrt{e} + \frac{1}{\sqrt{e}} - 2\right]$$

$$= 2\left[\sqrt{e} - 2 + \frac{1}{\sqrt{e}}\right]$$

$$\approx 0.5105$$

7 Evaluate: (a) $\int_2^3 \frac{1}{x-1}dx$ (b) $\int_0^3 \frac{2}{x+3}dx$ (c) $\int_{-2}^0 \frac{dx}{5+2x}$

(e) $\int_2^3 \left(x+\frac{1}{x-1}\right)dx$ (f) $\int_1^2 \left(x-\frac{1}{x^2}\right)^2 dx$ (g) $\int_1^3 \left(e^x+\frac{1}{x}\right)dx$

a) $\int_2^3 \frac{1}{x-1}\,dx = \Big[\ln(x-1)\Big]_2^3 = \ln 2 - \ln 1 = \ln 2$

b) $\int_0^3 \frac{2}{x+3}\,dx = \Big[2\ln(x+3)\Big]_0^3 = 2\ln 6 - 2\ln 3 = 2\ln\frac{6}{3} = 2\ln 2 = \ln 4$

c) $\int_{-2}^0 \frac{dx}{5+2x} = \left[\frac{\ln(5+2x)}{2}\right]_{-2}^0 = \frac{\ln 5}{2} - \frac{\ln 1}{2} = \frac{\ln 5}{2} = \ln\sqrt{5}$

e) $\int_2^3 \left(x+\frac{1}{x-1}\right)dx = \left[\frac{x^2}{2} + \ln(x-1)\right]_2^3 = \frac{9}{2} + \ln 2 - (2 + \ln 1)$

$= \frac{5}{2} + \ln 2$

f) $\int_1^2 \left(x-\frac{1}{x^2}\right)^2 dx = \int_1^2 \left(x^2 - \frac{2x}{x^2} + \frac{1}{x^4}\right)dx = \int_1^2 \left(x^2 - \frac{2}{x} + \frac{1}{x^4}\right)dx$

$= \left[\frac{x^3}{3} - 2\ln x + \frac{x^{-4+1}}{(-4+1)}\right]_1^2 = \left[\frac{x^3}{3} - 2\ln x + \frac{x^{-3}}{(-3)}\right]_1^2$

$= \left(\frac{8}{3} - 2\ln 2 - \frac{1}{3\times 8}\right) - \left(\frac{1}{3} - \frac{1}{3}\right) = \frac{21}{8} - \ln 4$

g) $\int_1^3 \left(e^x + \frac{1}{x}\right)dx = \Big[e^x + \ln x\Big]_1^3 = (e^3 + \ln 3) - (e + \ln 1)$

$= e^3 - e + \ln 3$

10 Find the area of the region enclosed by the curve $y = \frac{x}{x^2+1}$, the x-axis and the ordinates $x = 2$ and $x = 4$.

$$\int_2^4 \frac{x}{x^2+1}\,dx = \left[\frac{\ln(x^2+1)}{2}\right]_2^4$$

$$= \frac{\ln 17}{2} - \frac{\ln 5}{2}$$

$$= \frac{1}{2}\left(\ln 17 - \ln 5\right)$$

$$= \frac{1}{2}\ln\left(\frac{17}{5}\right) = \ln\sqrt{\frac{17}{5}}$$

14 $\int_0^1 \frac{e^x}{1+e^x}\,dx = \log_e c$. Find the value of c.

$$\int_0^1 \frac{e^x}{1+e^x}\,dx = \left[\ln(1+e^x)\right]_0^1 = \ln(1+e) - \ln(1+e^0)$$

$$= \ln(1+e) - \ln 2$$

$$= \ln\left(\frac{1+e}{2}\right)$$

So $c = \frac{1+e}{2}$

16 a) Differentiate $f(x) = x \ln x$

b) Find the area enclosed by the function $y = \ln x$ and the lines $y = 0$ and $x = a$ where $a > 1$

a) $u(x) = x \quad u'(x) = 1$

$v(x) = \ln x \quad v'(x) = \frac{1}{x}$

so $f'(x) = 1 \times \ln x + \frac{1}{x} x = \ln x + 1$

Hence $(x \ln x)' = \ln x + 1 \iff \ln x = (x \ln x)' - 1$

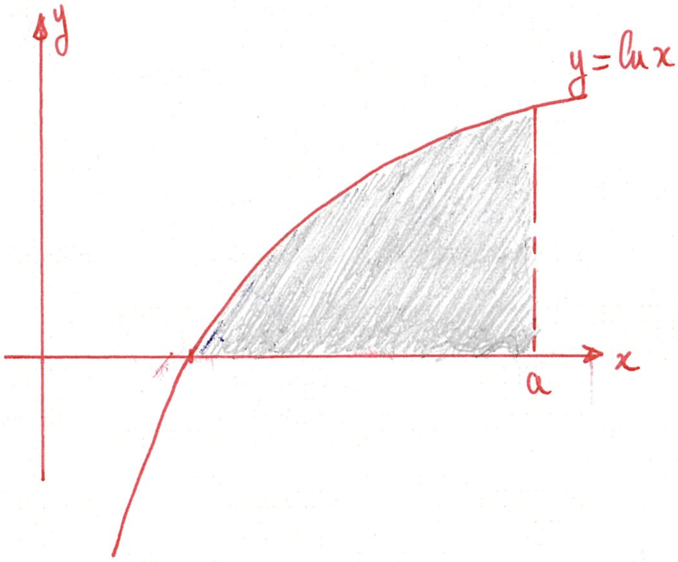
b) $\text{Area} = \int_1^a \ln x \, dx$

$\text{Area} = \int_1^a \left[(x \ln x)' - 1\right] dx$

$\text{Area} = \left[x \ln x\right]_1^a - \left[x\right]_1^a$

$\text{Area} = a \ln a - 1 \times \ln 1 - (a - 1)$

$\phantom{\text{Area}} = a \ln a - a + 1$



17 **(a)** Find $\frac{d}{dx}(\log_e(\cos x))$.

(b) Find the area enclosed by the curve $y = \tan x$, the x-axis and the ordinate $x = \frac{\pi}{3}$.

a) $\frac{d}{dx}\left(\ln(\cos x)\right) = \frac{1}{\cos x} \times (-\sin x) = -\tan x$ (Chain rule)

b) $\int_0^{\pi/3} \tan x \, dx = \int_0^{\pi/3} \left(-\frac{d}{dx}\left(\ln(\cos x)\right)\right) dx$

$= \int_{\pi/3}^{0} \left(\ln(\cos x)\right)' dx$

$= \left[\ln(\cos x)\right]_{\pi/3}^{0} = \ln(\cos 0) - \ln(\cos \pi/3)$

$= \ln 1 - \ln \frac{1}{2} = -\ln \frac{1}{2} = -\ln 2^{-1} = \ln 2$