FURTHER TRIGONOMETRY - CHAPTER REVIEW

1 Simplify: **(a)** $\frac{1-t^2}{1+t^2}$, where $t = \tan\frac{\theta}{2}$ **(b)** $\frac{\tan\theta - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{6}\tan\theta}$ **(c)** $\frac{\sin 2\theta - \sin\theta}{\cos 2\theta - \cos\theta + 1}$

a) $\frac{1-t^2}{1+t^2} = \cos A$ when $t = \tan \theta/2$

b) $\frac{\tan\theta - \tan\pi/6}{1+\tan\pi/6\,\tan\theta} = \tan\left(\theta - \frac{\pi}{6}\right)$

c) $\frac{\sin 2\theta - \sin\theta}{\cos 2\theta - \cos\theta + 1} = \frac{2\sin\theta\cos\theta - \sin\theta}{2\cos^2\theta - 1 - \cos\theta + 1}$

$= \frac{\sin\theta\,[2\cos\theta - 1]}{\cos\theta\,[2\cos\theta - 1]}$

$= \frac{\sin\theta}{\cos\theta} = \tan\theta$

2 Solve $2\tan 2x - 1 = 0$ for $0° < x < 360°$.

$\Leftrightarrow \tan 2x = \frac{1}{2}$ so $2x = \tan^{-1}\left(\frac{1}{2}\right) + n\pi$

$x = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) + n \times \frac{180}{2} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) + 90n$

$n = 0 \quad x = 13°.28$

$n = 1 \quad x = 103°.28$

$n = 2 \quad x = 193°.28$

$n = 3 \quad x = 283°28$

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3 Simplify:

(a) $\sin(\theta+\phi)\cos\phi-\cos(\theta+\phi)\sin\phi$ **(b)** $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$ **(c)** $\sin x\cos x\cos 2x\cos 4x$

a) $\sin(\theta+\phi)\cos\phi-\cos(\theta+\phi)\sin\phi=\sin[(\theta+\phi)-\phi]=\sin\theta$

b) $\dfrac{2\tan\theta/2}{1-\tan^2\theta/2}=\tan\left(2\times\dfrac{\theta}{2}\right)=\tan\theta$

c) $\sin x\cos x\cos 2x\cos 4x=\dfrac{\sin 2x}{2}\cos 2x\cos 4x$

$=\dfrac{1}{2}\left[\sin 2x\cos 2x\right]\cos 4x$

$=\dfrac{1}{2}\,\dfrac{\sin 4x}{2}\cos 4x$

$=\dfrac{1}{4}\left[\sin 4x\cos 4x\right]$

$=\dfrac{1}{4}\,\dfrac{\sin 8x}{2}=\dfrac{\sin 8x}{8}$

4 **(a)** Show that $\cos(A+B)=\cos A\cos B(1-\tan A\tan B)$.

(b) Suppose that $0<A<\frac{\pi}{2}$ and $0<B<\frac{\pi}{2}$. Show by deduction that if $\tan A\tan B=1$ then $A+B=\frac{\pi}{2}$.

a) $\cos A\cos B[1-\tan A\tan B]=\cos A\cos B\left[1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}\right]=\cos A\cos B-\sin A\sin B$
$=\cos(A+B)$

b) if $\tan A\tan B=1$, then $\tan A\tan B-1=0$,
hence $\cos(A+B)=0$.

If $0<A<\frac{\pi}{2}$ and $0<B<\frac{\pi}{2}$, that means that $A+B=\frac{\pi}{2}$

5 Show that: (a) $\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta-\tan\theta$ (b) $\tan^2\theta=\dfrac{1-\cos2\theta}{1+\cos2\theta}$, given that $\cos2\theta\neq-1$.

a) $$\frac{\cos\theta}{1+\sin\theta}=\frac{\cos\theta\,(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}=\frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}=\frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$$

So $$\frac{\cos\theta}{1+\sin\theta}=\frac{1-\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}=\sec\theta-\tan\theta$$

b) $$\frac{1-\cos2\theta}{1+\cos2\theta}=\frac{1-(1-2\sin^2\theta)}{1+2\cos^2\theta-1}$$

$$\text{———}=\frac{2\sin^2\theta}{2\cos^2\theta}$$

$$\text{———}=\frac{\sin^2\theta}{\cos^2\theta}$$

$$\text{———}=\left(\frac{\sin\theta}{\cos\theta}\right)^2=\tan^2\theta$$

6 Use the expansion of tan $2A$ to show that the exact value of $\tan 22.5° = \sqrt{2} - 1$. Hence find the exact value of $\tan 11.25°$.

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

For $2A = 45°$ $\quad \tan 2A = 1$

$$\frac{2 \tan 22.5}{1 - \tan^2 22.5} = 1 \iff \tan^2 22.5 + 2 \tan 22.5 - 1 = 0$$

we do a change of variable $x = \tan 22.5$

The equation becomes $x^2 + 2x - 1 = 0$

$\Delta = 4 - 4 \times (-1) = 8$ $\quad$ So $x = \dfrac{-2 + 2\sqrt{2}}{2} = \sqrt{2} - 1$

(the other solution is impossible as it's negative, whereas $\tan 22.5 > 0$)

So $\tan 22.5 = \sqrt{2} - 1$

Likewise $\tan 22.5 = \dfrac{2 \tan 11.25}{1 - \tan^2 11.25} = (\sqrt{2} - 1)$

So $\tan^2(11.25) \times [\sqrt{2} - 1] + 2 \tan(11.25) + 1 - \sqrt{2} = 0$

let $x = \tan(11.25)$ the equation becomes: $x^2(\sqrt{2}-1) + 2x + 1 - \sqrt{2} = 0$

$\Delta = 4 - 4(1-\sqrt{2})(\sqrt{2}-1) = 4 + 4(2 - 2\sqrt{2} + 1) = 16 - 8\sqrt{2}$

$$\tan(11.25) = \frac{-2 + \sqrt{16 - 8\sqrt{2}}}{2(\sqrt{2} - 1)} = \frac{-2 + 2\sqrt{4 - 2\sqrt{2}}}{2(\sqrt{2} - 1)}$$

$$\therefore \tan(11.25) = \frac{\sqrt{4 - 2\sqrt{2}} - 1}{\sqrt{2} - 1}$$

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7 Solve the following equations for $0 \le x \le \pi$.

(a) $\cos 3x = \cos 2x \cos x$ **(b)** $\cos 3x + \cos 5x + \cos 7x = 0$

a) $\iff \cos(x+2x) = \cos 2x \cos x$

$\iff \cos 2x \cos x - \sin 2x \sin x = \cos 2x \cos x$

$\iff -\sin 2x \sin x = 0$

$\iff 2 \sin x \cos x \sin x = 0$

$\iff \sin^2 x \cos x = 0$ so either $\sin x = 0$ i.e. $x = 0$, or π

OR $\cos x = 0$, i.e. $x = \pi/2$

$\therefore$ 3 solutions: $x = 0, \frac{\pi}{2}, \pi$

b) As $2 \cos A \cos B = \cos(A-B) + \cos(A+B)$

if $\begin{cases} A-B = \alpha \\ A+B = \beta \end{cases}$ then $\begin{cases} \alpha+\beta = 2A \\ \beta-\alpha = 2B \end{cases} \Rightarrow \begin{cases} A = \frac{\alpha+\beta}{2} \\ B = \frac{\beta-\alpha}{2} \end{cases}$

So $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta-\alpha}{2}\right)$

Hence $\cos 3x + \cos 5x + \cos 7x = 0$

$\iff \cos 5x + \cos 3x + \cos 7x = 0$

$\iff \cos 5x + 2\cos\left(\frac{3x+7x}{2}\right)\cos\left(\frac{7x-3x}{2}\right) = 0$

$\iff \cos 5x + 2\cos 5x \cos 2x = 0$

$\iff \cos 5x \left[1 + 2\cos 2x\right] = 0$

So either $\cos 5x = 0$ i.e. $5x = \pm\frac{\pi}{2} + 2n\pi$, $x = \pm\frac{\pi}{10} + \frac{2n\pi}{5}$

$n=0$ gives $x = \frac{\pi}{10}$, $n=1$ gives $\frac{\pi}{2}$ or $\frac{3\pi}{10}$, $n=2$ gives $\frac{9\pi}{10}$ or $\frac{7\pi}{10}$

OR $\cos 2x = -1/2$, i.e. $2x = \pm\frac{2\pi}{3} + 2n\pi$, i.e. $x = \pm\frac{\pi}{3} + n\pi$

$n=0$ gives $x = \pi/3$, $n=1$ gives $2\pi/3$

Solutions are $\pi/10, \frac{3\pi}{10}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{7\pi}{10}, \frac{9\pi}{10}$

8 Solve for $-\pi \le x \le \pi$.

(a) $\cos x - \sin x = 1$ **(b)** $\sin 4x - \sin 2x = 0$ **(c)** $\cos x - \sqrt{3}\sin x = 1$

a) $\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x = \frac{\sqrt{2}}{2} \iff \cos\left(\frac{\pi}{4} + x\right) = \cos\frac{\pi}{4}$

So $\frac{\pi}{4} + x = \pm\frac{\pi}{4} + 2n\pi \iff x = \pm\frac{\pi}{4} - \frac{\pi}{4} + 2n\pi$

$n=0$ gives: $x = -\frac{\pi}{2}$ or $x = 0$

$n=1$ gives $x = -\frac{\pi}{2} + 2\pi$ or 2π both outside $[-\pi, \pi]$

So 2 solutions $x = 0$ or $x = -\pi/2$

b) $\sin 4x = \sin 2x$ so $4x = (-1)^n 2x + n\pi$

$\iff x\left[4 - (-1)^n \times 2\right] = n\pi \implies x = \frac{1}{(2-(-1)^n)} n\frac{\pi}{2}$

$n=0$ gives $x=0$ $n=1$ gives $x = \frac{1}{3}\times\frac{\pi}{2} = \frac{\pi}{6}$

$n=2$ gives $x = \pi$ $n=-1$ gives $x = -\frac{\pi}{2}\times\frac{1}{3} = -\frac{\pi}{6}$

$n=-2$ gives $x = -\pi$ So 5 solutions: $-\pi, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \pi$

c) $\iff \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \frac{1}{2}$

$\iff \cos\frac{\pi}{3}\cos x - \sin\frac{\pi}{3}\sin x = \frac{1}{2} \iff \cos\left(x + \frac{\pi}{3}\right) = \cos\frac{\pi}{3}$

So $x + \frac{\pi}{3} = \pm\frac{\pi}{3} + 2n\pi \iff x = \pm\frac{\pi}{3} - \frac{\pi}{3} + 2n\pi$

$n=0$ gives $-\frac{2\pi}{3}$ and 0

$n=1$ gives $\frac{4\pi}{3}$ and 2π

$n=-1$ gives -2π and $\frac{-8\pi}{3}$

So 2 solutions in the interval which are $-\frac{2\pi}{3}$, 0

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a) Show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

b) Using an appropriate substitution, and a), solve $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$

c) Show that $\tan\frac{\pi}{9} - \tan\frac{2\pi}{9} + \tan\frac{4\pi}{9} = 3\sqrt{3}$

d) Show that $\tan^2\frac{\pi}{9} + \tan^2\frac{2\pi}{9} + \tan^2\frac{4\pi}{9} = 33$

a) $\tan(3\theta) = \tan(2\theta + \theta) = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \tan\theta \times \frac{2\tan\theta}{1-\tan^2\theta}} = \frac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta) - 2\tan^2\theta}$$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$\therefore \quad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

b) let $x = \tan\theta$. The equation becomes:

$$\tan^3\theta - 3\sqrt{3}\tan^2\theta - 3\tan\theta + \sqrt{3} = 0$$

$$\Leftrightarrow \quad -3\sqrt{3}\tan^2\theta + \sqrt{3} = 3\tan\theta - \tan^3\theta$$

$$\Leftrightarrow \quad \sqrt{3}\left[1 - 3\tan^2\theta\right] = 3\tan\theta - \tan^3\theta$$

$$\Leftrightarrow \quad \sqrt{3} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan 3\theta \quad \text{(from a))}$$

So $\tan 3\theta = \sqrt{3} = \tan\frac{\pi}{3}$

General solution is $3\theta = \frac{\pi}{3} + n\pi$

$$\Leftrightarrow \quad \theta = \frac{\pi}{9} + n\frac{\pi}{3}$$

$n=0$ gives $\theta = \frac{\pi}{9}$ so $x = \tan\frac{\pi}{9}$ is a solution of the cubic

$n=1$ gives $\theta = \frac{\pi}{9} + \frac{\pi}{3} = \frac{4\pi}{9}$ so $x = \tan\frac{4\pi}{9}$ is also a solution

$n=2$ gives $\theta = \frac{\pi}{9} + \frac{2\pi}{3} = \frac{7\pi}{9}$ so $x = \tan\frac{7\pi}{9}$ is the 3rd and last solution of the cubic and they're all different

c) We know that if α, β and γ are solutions of a cubic equation, then $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{3\sqrt{3}}{1}$ in that case.

Hence, we must have $\tan\frac{\pi}{9} + \tan\frac{4\pi}{9} + \tan\frac{7\pi}{9} = 3\sqrt{3}$

We note that $\tan\frac{7\pi}{9} = \tan\left(\pi - \frac{2\pi}{9}\right) = \tan\left(\frac{-2\pi}{9}\right) = -\tan\left(\frac{2\pi}{9}\right)$

$$\therefore \quad \tan\left(\frac{\pi}{9}\right) - \tan\left(\frac{2\pi}{9}\right) + \tan\left(\frac{4\pi}{9}\right) = 3\sqrt{3}$$

d) $$\underbrace{(\alpha+\beta+\gamma)^2}_{=-\frac{b}{a}=3\sqrt{3}} = \alpha^2+\beta^2+\gamma^2 + 2\underbrace{(\alpha\beta+\alpha\gamma+\beta\gamma)}_{=\frac{c}{a} = \frac{-3}{1} = -3}$$

$$\therefore \quad \alpha^2+\beta^2+\gamma^2 = (3\sqrt{3})^2 - 2\times(-3)$$

$$= 9\times 3 + 6$$

$$\therefore \quad \alpha^2+\beta^2+\gamma^2 = 33$$

So $\left[\tan\left(\frac{\pi}{9}\right)\right]^2 + \left[-\tan\left(\frac{2\pi}{9}\right)\right]^2 + \left[\tan\left(\frac{4\pi}{9}\right)\right]^2 = 33$

$$\therefore \quad \tan^2\left(\frac{\pi}{9}\right) + \tan^2\left(\frac{2\pi}{9}\right) + \tan^2\left(\frac{4\pi}{9}\right) = 33$$