INTEGRATION OF TRIGONOMETRIC FUNCTIONS

The simplest integrals involving the trigonometric functions are:

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C \qquad \int \sin ax\,dx = -\frac{1}{a}\cos ax + C \qquad \int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

The identity $\tan^2 x = \sec^2 x - 1$ allows $\int \tan^2 x\,dx$ to be found.

The double-angle results are:

- $\sin 2x = 2\sin x\cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x,$

which can be rewritten as $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$.

These results can be used to reduce trigonometric expressions to the simpler forms given as the standard integrals, which allows you to find integrals such as:

$$\int \sin x\cos x\,dx = \frac{1}{2}\int \sin 2x\,dx \qquad \int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx \qquad \int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$

Integrals $\int \sin^m x\,dx$ and $\int \cos^m x\,dx$, m an even positive integer

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

Thus you can write:

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right)$$
$$= \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)$$
$$= \frac{1}{8}(3+4\cos 2x+\cos 4x)$$

Example 1

Find:

(a) $\int \cos^2 x\,dx$ **(b)** $\int \sin^4 x\,dx$ **(c)** $\int \sin^2 2x\,dx$ **(d)** $\int \sin^2 x\cos^2 x\,dx$

Solution

(a) $\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

(b) $\sin^4 x = \frac{1}{4}(1-\cos 2x)^2$ as $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right)$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)$$

$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

Hence: $\int \sin^4 x\,dx = \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\,dx$

$$= \frac{1}{8}\left(3x - 2\sin 2x + \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

(c) $\int \sin^2 2x\,dx = \frac{1}{2}\int (1-\cos 4x)\,dx = \frac{x}{2} - \frac{1}{8}\sin 4x + C$

(d) As $\sin x\cos x = \frac{1}{2}\sin 2x$:

$$\sin^2 x\cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{8}(1-\cos 4x)$$

Hence: $\int \sin^2 x\cos^2 x\,dx = \frac{1}{8}\int (1-\cos 4x)\,dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

Alternatively: $\sin^2 x\cos^2 x = \sin^2 x(1-\sin^2 x) = \sin^2 x - \sin^4 x$

Now $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

and $\int \sin^4 x\,dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ from part **(b)**

Hence: $\int \sin^2 x\cos^2 x\,dx = \int \sin^2 x\,dx - \int \sin^4 x\,dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

Use of $\int f(u)\,du$, $u = g(x)$ for trigonometric integrals

Example 2

Find: **(a)** $\int \cos x \sin x \, dx$ **(b)** $\int \cos x \sin^3 x \, dx$ **(c)** $\int x \cos x^2 \, dx$

Solution

(a) Let $u = \sin x$ so that $du = \cos x \, dx$

$$\int \cos x \sin x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{1}{2}\sin^2 x + C$$

This method may be used instead of writing the integrand in terms of sin $2x$ as mentioned earlier.

(b) Let $u = \sin x$ so that $du = \cos x \, dx$:

$$\int \cos x \sin^3 x \, dx = \int u^3 \, du = \frac{1}{4}u^4 + C$$
$$= \frac{1}{4}\sin^4 x + C$$

Alternatively:

$\int \cos x \sin^3 x \, dx = \int \sin^3 x \cos x \, dx$ is of the form $\int [f(x)]^n f'(x) \, dx = \frac{1}{n+1}[f(x)]^{n+1} + C$

$\therefore$ This integral is the reverse of the function-of-a-function rule, so:

$$\int \cos x \sin^3 x \, dx = \frac{1}{4}\sin^4 x + C$$

(c) Let $u = x^2$ so that $du = 2x \, dx$:

$$\int x \cos x^2 \, dx = \frac{1}{2}\int \cos u \, du = \frac{1}{2}\sin u + C$$
$$= \frac{1}{2}\sin x^2 + C$$

Example 3

Find: **(a)** $\int \cos^3 x\,dx$ **(b)** $\int \sin^3 x\,dx$ **(c)** $\int \sin^2 x\cos^3 x\,dx$

Solution

(a)

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x\cos x\,dx\\ &= \int (1-\sin^2 x)\cos x\,dx\\ &= \int (\cos x - \sin^2 x\cos x)\,dx\end{aligned}$$

Now use $\int [f(x)]^2 f'(x)\,dx = \frac{1}{3}[f(x)]^3 + C$ where $f(x) = \sin x$ so that $f'(x) = \cos x$

$$\therefore \int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$$

(b)

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin^2 x\sin x\,dx\\ &= \int (1-\cos^2 x)\sin x\,dx\\ &= \int (\cos^2 x - 1)(-\sin x)\,dx\\ &= \int (-\sin x\cos^2 x + \sin x)\,dx\\ &= \frac{1}{3}\cos^3 x - \cos x + C\end{aligned}$$

Alternatively: Let $u = \cos x$ so that $du = -\sin x\,dx$:

$$\begin{aligned}\int \sin^3 x\,dx &= \int (u^2-1)\,du\\ &= \frac{1}{3}u^3 - u + C\\ &= \frac{1}{3}\cos^3 x - \cos x + C\end{aligned}$$

(c)

$$\begin{aligned}\int \sin^2 x\cos^3 x\,dx &= \int (1-\cos^2 x)\cos^3 x\,dx\\ &= \int (\cos^3 x - \cos^5 x)\,dx\end{aligned}$$

Now $\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$ from **(a)**

and

$$\begin{aligned}\int \cos^5 x\,dx &= \int \cos^4 x\cos x\,dx\\ &= \int (1-\sin^2 x)^2\cos x\,dx\\ &= \int (1 - 2\sin^2 x + \sin^4 x)\cos x\,dx\\ &= \int (\cos x - 2\sin^2 x\cos x + \sin^4 x\cos x)\,dx\\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C\end{aligned}$$

Thus:

$$\begin{aligned}\int (\cos^3 x - \cos^5 x)\,dx &= \sin x - \frac{1}{3}\sin^3 x - \left(\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right) + C\\ &= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C\end{aligned}$$

Alternatively: $\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x\cos^2 x\cos x\,dx$

Let $u = \sin x$ so that $du = \cos x\,dx$ and $\cos^2 x = 1 - u^2$:

$$\begin{aligned}\int \sin^2 x\cos^3 x\,dx &= \int u^2(1-u^2)\,du\\ &= \int (u^2 - u^4)\,du\\ &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C\\ &= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C\end{aligned}$$

Example 4

Evaluate: **(a)** $\int_0^{\frac{\pi}{4}} \cos^2 2x\,dx$ **(b)** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x\,dx$ **(c)** $\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx$

Solution

(a) $\int_0^{\frac{\pi}{4}} \cos^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 4x)\,dx$

$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left(\frac{\pi}{4} + 0 - 0\right)$

$= \frac{\pi}{8}$

(b) Let $u = \cos x$, $du = -\sin x\,dx$.

Calculate the new limits of the definite integral, $x = \frac{\pi}{3}$: $u = \cos\frac{\pi}{3} = \frac{1}{2}$ $x = \frac{\pi}{2}$: $u = \cos\frac{\pi}{2} = 0$

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x\,dx = -\int_{\frac{1}{2}}^{0} u^3\,du$

$= -\left[\frac{u^4}{4}\right]_{\frac{1}{2}}^{0}$

$= \frac{1}{64}$

or

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x\,dx = -\frac{1}{4}\left[\cos^4 x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= -\frac{1}{4}\left(0 - \frac{1}{16}\right)$

$= \frac{1}{64}$

(c) $\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx = \int_{\frac{3\pi}{4}}^{\pi} \sin^2 x \sin x\,dx = \int_{\frac{3\pi}{4}}^{\pi} (1-\cos^2 x)\sin x\,dx$

Let $u = \cos x$, $du = -\sin x\,dx$.

$x = \frac{3\pi}{4}$: $u = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ $x = \pi$: $u = \cos\pi = -1$

$\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx = -\int_{-\frac{1}{\sqrt{2}}}^{-1} (1-u^2)\,du$

$= \int_{-\frac{1}{\sqrt{2}}}^{-1} (u^2-1)\,du$

$= \left[\frac{u^3}{3} - u\right]_{-\frac{1}{\sqrt{2}}}^{-1}$

$= \left(-\frac{1}{3} + 1 - \left(-\frac{1}{6\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right)$

$= \frac{2}{3} - \frac{5\sqrt{2}}{12}$

or

$\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx = \int_{\frac{3\pi}{4}}^{\pi} (\sin x - \sin x \cos^2 x)\,dx$

$= \left[-\cos x + \frac{1}{3}\cos^3 x\right]_{\frac{3\pi}{4}}^{\pi}$

$= 1 - \frac{1}{3} - \left(\frac{1}{\sqrt{2}} + \frac{1}{3}\times\left(-\frac{1}{\sqrt{2}}\right)^3\right)$

$= \frac{2}{3} - \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)$

$= \frac{2}{3} - \frac{5}{6\sqrt{2}} = \frac{2}{3} - \frac{5\sqrt{2}}{12}$