PARAMETRIC FORM OF A FUNCTION OR RELATION

It is often useful in mathematics to express two related variables (e.g. x and y) in terms of a third variable (e.g. t or θ), so that, for example: $x = f(t)$, $y = g(t)$ or $x = f(\theta)$, $y = g(\theta)$

Equations like these are called **parametric equations** and the third variable (e.g. t or θ) is called the **parameter**.

For example, recall that the functions cosine and sine can be defined as the x- and y-coordinates respectively of a point on the unit circle $x^2 + y^2 = 1$. Thus the unit circle can be represented by the parametric equations:

$x = \cos\theta,\ y = \sin\theta$

where θ is the parameter. When the unit circle is described by the equation $x^2 + y^2 = 1$, it is said to be in **Cartesian** form.

Example 16

Find the Cartesian equation of the curve and describe it in words, given the parametric equations:

(a) $x = t, y = t + 1$ **(b)** $x = 2t - 1, y = 3t + 2$.

Solution

(a) $x = t, y = t + 1$

Make t the subject of the equation in x: $t = x$

Substitute in the equation for y: $y = x + 1$

The parametric equations represent a straight line with gradient 1 and y-intercept 1.

(b) $x = 2t - 1, y = 3t + 2$

Make t the subject of the equation in x: $2t = x + 1$

$$t = \frac{x+1}{2}$$

Substitute in the equation for y: $y = 3 \times \frac{x+1}{2} + 2$

$$2y = 3x + 3 + 4$$

$$3x - 2y + 7 = 0$$

The parametric equations represent a straight line with gradient 1.5 and y-intercept 3.5.

Example 17

Find the Cartesian equation of the curve whose parametric equations are $x = 1 + t, y = t^2$.

Solution

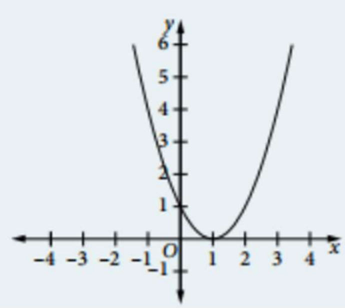
$x = 1 + t$ [1]

$y = t^2$ [2]

From [1]: $t = x - 1$

Substitute into [2]: $y = (x - 1)^2$

Hence the Cartesian equation is $y = (x - 1)^2$ and the graph is the parabola shown.



Example 18

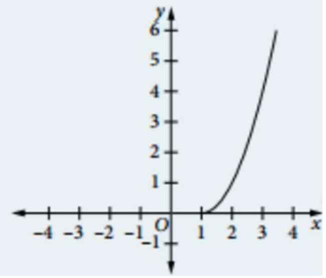
Find the Cartesian equation of the curve whose parametric equations are $x = 1 + t$, $y = t^2$, $t \geq 0$.

Solution

As in Example 17, these parametric equations give the Cartesian equation $y = (x - 1)^2$, but there is now also the condition $t \geq 0$.

$x = 1 + t$ and $t \geq 0$, so the condition is equivalent to $x \geq 1$.

Hence the Cartesian equation is $y = (x - 1)^2$ with the domain restricted to $x \geq 1$, as shown.



Example 19

Find the Cartesian equation of the curve whose parametric equations are given by $x = 2\sin\theta$, $y = 2\cos\theta$. Describe the curve in words and sketch its graph.

Solution

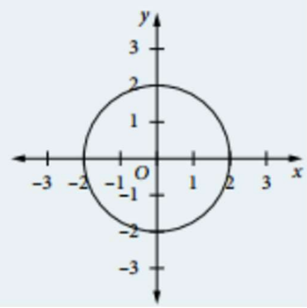
Recall the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$.

$\sin\theta = \frac{x}{2}$, so $\sin^2\theta = \frac{x^2}{4}$. $\cos\theta = \frac{y}{2}$, so $\cos^2\theta = \frac{y^2}{4}$.

Hence, using the identity: $\frac{x^2}{4} + \frac{y^2}{4} = 1$

or $x^2 + y^2 = 4$

The curve is a circle with centre at the origin and radius 2.



Example 20

Write each Cartesian equation in parametric form.

(a) $3x + y - 3 = 0$ **(b)** $x^2 = 4(y - 3)$ **(c)** $(x-1)^2 + (y+2)^2 = 9$

Solution

There may be more that one set of parametric equations for each Cartesian equation, depending on how the parameter is defined for x and y.

(a) **Method 1**

Rewrite the equation: $y = 3 - 3x$

Let $t = x$: $y = 3 - 3t$

The parametric equations are $x = t$ and $y = 3 - 3t$.

Method 2

Rewrite the equation: $y = 3(1 - x)$

Let $t = 1 - x$: $y = 3t$

The parametric equations are $x = 1 - t$ and $y = 3t$.

(b) Rewrite the equation: $\left(\frac{x}{2}\right)^2 = y - 3$

Method 1

Let $t = \frac{x}{2}$: $t^2 = y - 3$

$x = 2t \qquad y = t^2 + 3$

The parametric equations are $x = 2t$ and $y = t^2 + 3$.

Method 2

Rewrite the equation: $x^2 = 4(y - 3)$

Let $t = x$: $t^2 = 4(y - 3)$

$y = \frac{t^2}{4} + 3$

The parametric equations are $x = t$ and $y = \frac{t^2}{4} + 3$.

(c) The equation is the sum of two squares, which suggests that the identity $\sin^2\theta + \cos^2\theta = 1$ may be useful.

Rewrite the equation: $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{9} = 1$

Method 1

Let $\frac{x-1}{3} = \sin\theta$ and $\frac{y+2}{3} = \cos\theta$

$x = 3\sin\theta + 1 \qquad y = 3\cos\theta - 2$

The parametric equations are $x = 3\sin\theta + 1$ and $y = 3\cos\theta - 2$.

Method 2

Swapping the position of $\sin\theta$ and $\cos\theta$ would give different parametric equations.

Parametric equations of the parabola

The parabola $x^2 = 4ay$ can be represented by the parametric equations: $x = 2at$, $y = at^2$

This can be verified by eliminating the parameter:

$x = 2at \qquad [1]$

$y = at^2 \qquad [2]$

From [1]: $t = \frac{x}{2a}$

Substitute into [2]: $y = a\left(\frac{x}{2a}\right)^2$

$x^2 = 4ay$

The point $P(2at, at^2)$ on the parabola is the variable point that depends on the value of t, so it is frequently called 'the point t'.

Example 21

Sketch the graph of each curve from its parametric equations.

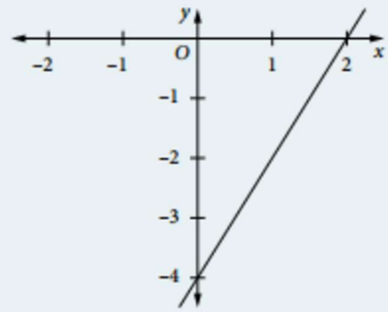
(a) $x = t + 2,\ y = 2t$ **(b)** $x = 2t,\ y = 2t^2$ **(c)** $x = 2\sin\theta,\ y = 2\cos\theta$

Solution

Either use graphing software or draw up a table of values and plot points.

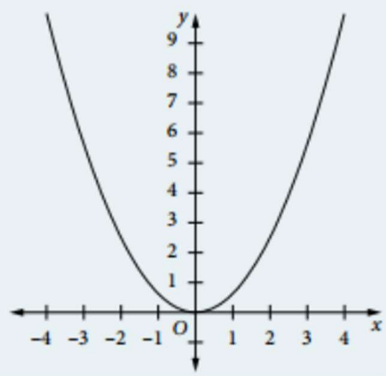
(a)

t	−2	−1	0
x	0	1	2
y	−4	−2	0



(b)

t	−2	−1	0	1	2
x	−4	−2	0	2	4
y	8	2	0	2	8



(c)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$x = 2\sin\theta$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	0.5	0
$y = 2\cos\theta$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	−1	$-\sqrt{2}$	$-\sqrt{3}$	−1

This table gives the right half of the graph of the relation. By changing the signs on values as the quadrant for θ changes, the left half may be graphed.