# INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

In the Mathematics Extension 1 course, you established the derivatives of the inverse trigonometric functions (see *New Senior Mathematics Extension 1 for Years 11 & 12*, Chapter 11). These lead to the following integrals.

- $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \quad \text{for } -a < x < a$
- $\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\frac{x}{a} + C \quad \text{for all } x$
- $\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\frac{x}{a} + C \quad \text{for } -a < x < a$

The integrals in the following example were first considered in the Mathematics Extension 1 course.

## Example 5

Find: **(a)** $\int \frac{dx}{\sqrt{4 - x^2}}$ **(b)** $\int \frac{-1}{\sqrt{9 - x^2}}\,dx$ **(c)** $\int \frac{2}{4 + x^2}\,dx$ **(d)** $\int \frac{dx}{\sqrt{3 - x^2}}$

**(e)** $\int \frac{dx}{\sqrt{1 - 9x^2}}$ **(f)** $\int \frac{dx}{1 + 4x^2}$ **(g)** $\int \frac{2}{\sqrt{4 - 25x^2}}\,dx$

## Solution

**(a)** $\int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1}\frac{x}{2} + C$

**(b)** $\int \frac{-1}{\sqrt{9 - x^2}}\,dx = \cos^{-1}\frac{x}{3} + C$

**(c)** $\int \frac{2}{4 + x^2}\,dx = \tan^{-1}\frac{x}{2} + C$

**(d)** $\int \frac{dx}{\sqrt{3 - x^2}} = \sin^{-1}\frac{x}{\sqrt{3}} + C$

**(e)** Write: $\sqrt{1 - 9x^2} = \sqrt{9\left(\frac{1}{9} - x^2\right)} = 3\sqrt{\left(\frac{1}{3}\right)^2 - x^2}$

Thus: $\int \frac{dx}{\sqrt{1 - 9x^2}} = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}\,dx = \frac{1}{3}\sin^{-1}\frac{x}{\left(\frac{1}{3}\right)} + C = \frac{1}{3}\sin^{-1}3x + C$

**(f)** Write: $1 + 4x^2 = 4\left(\frac{1}{4} + x^2\right) = 4\left(\left(\frac{1}{2}\right)^2 + x^2\right)$

Thus: $\int \frac{dx}{1 + 4x^2} = \frac{1}{4}\int \frac{1}{\left(\frac{1}{2}\right)^2 + x^2}\,dx = \frac{1}{2}\int \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2 + x^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{\left(\frac{1}{2}\right)} + C = \frac{1}{2}\tan^{-1}2x + C$

**(g)** Write: $\sqrt{4 - 25x^2} = \sqrt{25\left(\frac{4}{25} - x^2\right)} = 5\sqrt{\left(\frac{2}{5}\right)^2 - x^2}$

Thus: $\int \frac{2}{\sqrt{4 - 25x^2}}\,dx = \frac{2}{5}\int \frac{1}{\sqrt{\left(\frac{2}{5}\right)^2 - x^2}}\,dx = \frac{2}{5}\sin^{-1}\frac{x}{\left(\frac{2}{5}\right)} + C = \frac{2}{5}\sin^{-1}\frac{5x}{2} + C$

## Integration by substitution

The integrals in the previous example were found by the standard trigonometric integrals. They can be found explicitly by the method of change of variable, also called 'integration by substitution', i.e. the process:

$$\int f(x)\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

An expression like $\sqrt{a^2 - x^2}$ can be integrated with a substitution of $x = a\sin\theta$ or $x = a\cos\theta$, while an expression like $a^2 + x^2$ can be integrated with a substitution of $x = a\tan\theta$, as shown in the following examples.

### Example 6

Find: **(a)** $\int \frac{dx}{\sqrt{4-x^2}}$ **(b)** $\int \frac{-dx}{\sqrt{1-9x^2}}$ **(c)** $\int \frac{dx}{4+x^2}$

### Solution

**(a)** $\sqrt{4-x^2} = \sqrt{a^2-x^2}$ with $a = 2$, so use the substitution $x = 2\sin\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

($x \neq \pm\frac{\pi}{2}$ as these values make the denominator of the integrand zero)

Thus: $\frac{dx}{d\theta} = 2\cos\theta,\ dx = 2\cos\theta\,d\theta$

and $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$ as $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (i.e. $\cos\theta > 0$)

Hence:
$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta\,d\theta}{2\cos\theta} = \int d\theta$$
$$= \theta + C$$
$$= \sin^{-1}\frac{x}{2} + C$$

**(b)** $\sqrt{1-9x^2} = 3\sqrt{\frac{1}{9}-x^2} = 3\sqrt{\left(\frac{1}{3}\right)^2 - x^2} = 3\sqrt{a^2-x^2}$, so put $a = \frac{1}{3}$ and use the substitution $x = \frac{1}{3}\cos\theta$ for $0 < \theta < \pi$ ($x \neq 0, \pi$ as these values make the denominator of the integrand zero).

Thus: $\frac{dx}{d\theta} = -\frac{1}{3}\sin\theta,\ dx = -\frac{1}{3}\sin\theta\,d\theta$

and $\sqrt{1-9x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$

Hence:
$$\int \frac{-dx}{\sqrt{1-9x^2}} = \int \frac{-1}{\sin\theta}\times\left(-\frac{1}{3}\sin\theta\right)d\theta = \frac{1}{3}\int d\theta$$
$$= \frac{\theta}{3} + C$$
$$= \frac{1}{3}\cos^{-1}3x + C$$

**(c)** $4 + x^2 = a^2 + x^2$ with $a = 2$, so use the substitution $x = 2\tan\theta$

Thus: $\frac{dx}{d\theta} = 2\sec^2\theta,\ dx = 2\sec^2\theta\,d\theta$

and $4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$

Hence:
$$\int \frac{dx}{4+x^2} = \int \frac{1}{4\sec^2\theta}\times 2\sec^2\theta\,d\theta = \frac{1}{2}\int d\theta$$
$$= \frac{\theta}{2} + C$$
$$= \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

In Example **6(a)**, the restriction that $\theta$ is in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ has several implications.

It means that $\sqrt{4-x^2} = \sqrt{4\cos^2\theta}$ requires you to find the positive square root of the expression, because $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ implies $\cos\theta > 0$ and hence $\sqrt{4-x^2} = 2\cos\theta$.

If the restriction had been $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ then $\cos\theta < 0$ and so $\sqrt{4-x^2} = -2\cos\theta$.

The domain of the substitution should be chosen to coincide with the corresponding inverse function:

- for the substitution $x = a\sin\theta$, set $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- for the substitution $x = a\cos\theta$, set $0 < \theta < \pi$.

The endpoints of the domain are not included as $\frac{1}{\sqrt{a^2-x^2}}$ is undefined there.

For the substitution $x = \tan\theta$, no restriction on $\theta$ is required: $1 + x^2 > 0$ for all values of $x$, so the denominator of the integrand will never be zero for real values of $x$.

## Example 7

Find: **(a)** $\int \frac{dx}{(x+2)^2+9}$ **(b)** $\int \sqrt{4-x^2}\,dx$ for $|x| \le 2$ **(c)** $\int x\sqrt{4-x^2}\,dx$ for $|x| \le 2$

### Solution

**(a)** Let $(x+2)^2 + 9 = 9 + u^2$ where $u = x+2$. Thus $du = dx$.

Hence:
$$\int \frac{dx}{(x+2)^2+9} = \int \frac{du}{9+u^2} = \frac{1}{3}\int \frac{3}{9+u^2}\,du$$
$$= \frac{1}{3}\tan^{-1}\frac{u}{3} + C$$
$$= \frac{1}{3}\tan^{-1}\frac{x+2}{3} + C$$

**(b)** $\sqrt{4-x^2}$ suggests the substitution $x = 2\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

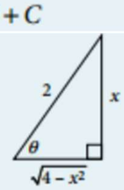
Note that $\theta = \pm\frac{\pi}{2}$ is allowed in this case as $\sqrt{4-x^2}$ does not occur in the denominator.

Let $x = 2\sin\theta$ so that $dx = 2\cos\theta\,d\theta$ and: $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

Hence:
$$\int \sqrt{4-x^2}\,dx = \int 2\cos\theta \times 2\cos\theta\,d\theta = 2\int 2\cos^2\theta\,d\theta$$
$$= 2\int (1+\cos 2\theta)\,d\theta$$
$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= 2\theta + \sin 2\theta + C$$
$$= 2\theta + 2\sin\theta\cos\theta + C$$

Now $x = 2\sin\theta$, so $\theta = \sin^{-1}\frac{x}{2}$ and $\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$$\therefore \int \sqrt{4-x^2}\,dx = 2\sin^{-1}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C \quad \text{for } |x| \le 2$$




**(c)** $x\sqrt{4-x^2}$ is of the form $f'(x)[f(x)]^n$, which suggests a substitution of the form $u = 4 - x^2$ rather than a trigonometric substitution.

Let $u = 4 - x^2$, so $du = -2x\,dx$, $x\,dx = -\frac{du}{2}$

Hence:
$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\,du$$
$$= -\frac{1}{3}u^{\frac{3}{2}} + C$$
$$= -\frac{1}{3}(4-x^2)^{\frac{3}{2}} + C \quad \text{for } |x| \le 2$$

The substitution $x = 2\sin\theta$ would have found the same answer, but it would have taken more steps.

## Example 8

Evaluate: **(a)** $\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}}$ **(b)** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}dx$ **(c)** $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **(d)** $\int_0^2 \sqrt{4-x^2}\,dx$

## Solution

**(a)** $\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}} = \left[\sin^{-1}\frac{x}{5}\right]_0^{2.5} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6}$

**Alternatively:** Use the substitution $x = 5\sin\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$:

$x = 5\sin\theta,\ dx = 5\cos\theta\, d\theta$: $\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\cos\theta$

Limits of integration are $x = 0$: $\theta = 0$ $x = 2.5$: $\sin\theta = \frac{1}{2}, \theta = \frac{\pi}{6}$

Hence: $\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}} = \int_0^{\frac{\pi}{6}} \frac{1}{5\cos\theta} \times 5\cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} d\theta = [\theta]_0^{\frac{\pi}{6}} = \frac{\pi}{6}$

**(b)** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}dx = \left[\cos^{-1}\frac{x}{2}\right]_{-\sqrt{3}}^{\sqrt{3}} = \cos^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{\pi}{6} - \frac{5\pi}{6} = -\frac{2\pi}{3}$

**Alternatively:** The substitution $x = 2\cos\theta$ could have been used.

**(c)** $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

**Alternatively:** The substitution $x = \tan\theta$ could have been used.

**(d)** Let $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$:

$x = 2\sin\theta,\ dx = 2\cos\theta\, d\theta$: $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

Limits of integration are $x = 0$: $\theta = 0$ $x = 2$: $\sin\theta = 1,\ \theta = \frac{\pi}{2}$

Hence: $\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{2}} 2\cos\theta \times 2\cos\theta\, d\theta$

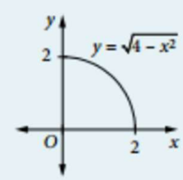
$$= 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$

$$= [2\theta + \sin 2\theta]_0^{\frac{\pi}{2}} = (\pi + \sin\pi) - (0 + \sin 0) = \pi$$

The graph of $y = \sqrt{4-x^2}$ for the domain $0 \le x \le 2$ is the quadrant of the circle $x^2 + y^2 = 4$ that is in the first quadrant. The formula for the area of a circle could have been used to evaluate the integral:

Area $= \frac{1}{4} \times \pi \times 2^2 = \pi$ (as $r = 2$)



## Sum or difference of two squares

All quadratic expressions of the form $ax^2 + bx + c$ can be expressed as the sum or the difference of two squares, depending on whether their discriminant $\Delta = b^2 - 4ac$ is positive or negative. This can be achieved by completing the square of the quadratic.

After the quadratic expression is written as the sum or difference of squares, its associated integral can usually be found by using an appropriate substitution.

### Example 9

Find: **(a)** $\int \frac{dx}{x^2+4x+13}$ **(b)** $\int \frac{dx}{\sqrt{9+16x-4x^2}}$ **(c** $\int \frac{dx}{\sqrt{2x-x^2}}$

#### Solution

**(a)** Completing the square:

$$x^2+4x+13 = x^2+4x+4+9$$
$$= (x+2)^2+9$$

Let $u = x+2$ so $du = dx$

Hence $(x+2)^2+9 = 9+u^2$:

$$\int \frac{dx}{x^2+4x+13} = \int \frac{1}{9+u^2}du$$
$$= \frac{1}{3}\tan^{-1}\frac{u}{3}+C$$
$$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+C$$

**(b)** Completing the square:

$$9+16x-4x^2 = -4\left[x^2-4x-\frac{9}{4}\right]$$
$$= -4\left[(x^2-4x+4)-4-\frac{9}{4}\right]$$
$$= 25-4(x-2)^2$$

Let $u = x-2$ so $du = dx$

Hence $25-4(x-2)^2 = 25-4u^2$:

$$\int \frac{dx}{\sqrt{9+16x-4x^2}} = \int \frac{du}{\sqrt{25-4u^2}}$$
$$= \frac{1}{2}\int \frac{du}{\sqrt{\frac{25}{4}-u^2}}$$
$$= \frac{1}{2}\sin^{-1}\frac{2u}{5}+C$$
$$= \frac{1}{2}\sin^{-1}\frac{2(x-2)}{5}+C$$

**(c)** Completing the square: $2x-x^2 = -[(x^2-2x+1)-1]$
$$= 1-(x-1)^2$$

Let $u = x-1$ so $du = dx$

Hence $1-(x-1)^2 = 1-u^2$:

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{du}{\sqrt{1-u^2}}$$
$$= \sin^{-1}u+C$$
$$= \sin^{-1}(x-1)+C$$