$r^4 e^{i4\theta} = (re^{i\theta}) = z^4 = 16i = 2^4 e^{i\pi/2}$

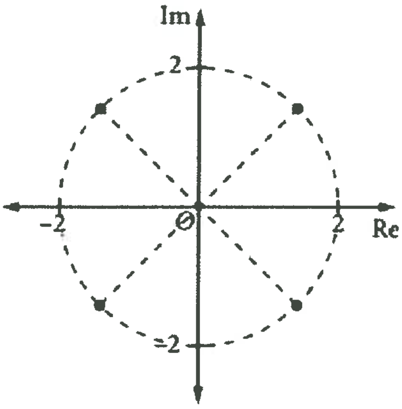
ROOTS OF COMPLEX NUMBERS

so $z = 2e^{i\left(\frac{\pi}{8} + \frac{2n\pi}{4}\right)}$

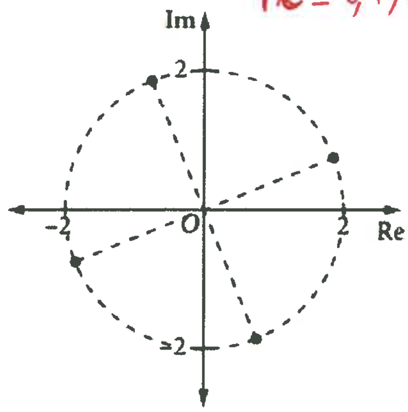
$n = 0, 1, 2, 3$

1 Which Argand diagram best shows the fourth roots of $16i$?

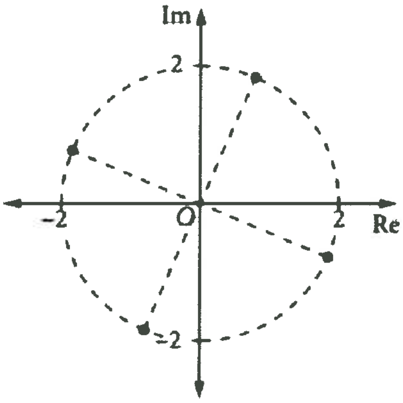
A



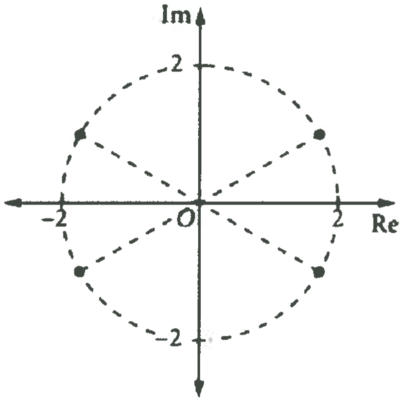
$\boxed{\textbf{B}}$



C



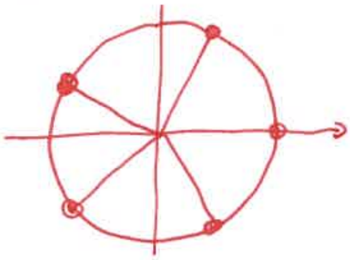
D



2 For each of the following, find the values of z (in mod–arg form) and plot them on the complex plane.

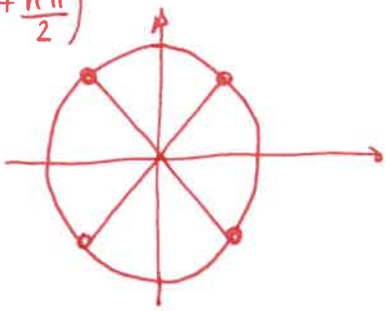
(a) $z^5 = 1$ (b) $z^4 + 1 = 0$ (c) $z^2 = i$ (d) $z^3 + 8i = 0$ (e) $z^4 = 8(\sqrt{3} + i)$ (f) $z^6 = i$

a) $z = e^{\frac{i}{5}(2n\pi)} = e^{i\frac{2n\pi}{5}}$



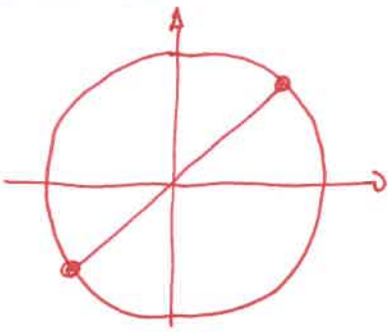
b) $z^4 = -1 = e^{i\pi}$ so $z = e^{i\left(\frac{\pi}{4} + \frac{2n\pi}{4}\right)}$

$z = e^{i\left(\pi/4 + \frac{n\pi}{2}\right)}$

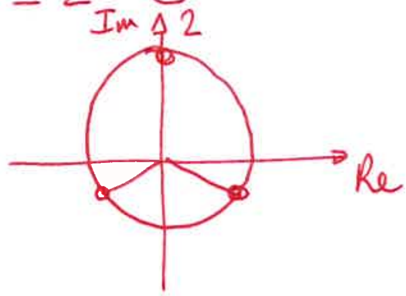


c) $z^2 = e^{i\pi/2}$ so $z = e^{i\left(\frac{\pi}{4} + \frac{2n\pi}{2}\right)}$

$z = e^{i(\pi/4 + n\pi)}$



d) $z^3 + 8i = 0 \iff z^3 = -8i = 2^3(-i) = 2^3 e^{-i\pi/2}$



so $z = 2e^{i\left(-\frac{\pi}{6} + \frac{2n\pi}{3}\right)}$

$n=0 \quad z = 2e^{-i\pi/6}$

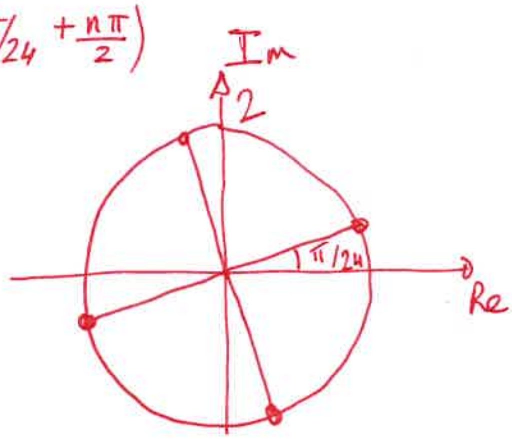
$n=1 \quad z = 2e^{i\pi/2} = 2i$

$n=2 \quad z = 2e^{i\left(-\pi/6 + \frac{4\pi}{3}\right)} = 2e^{\frac{i7\pi}{6}} = 2e^{-i5\pi/6}$

as the principal argument must be between $(-\pi)$ and $(+\pi)$

e) $z^4 = 8(\sqrt{3}+i) = 16\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2^4 e^{i\pi/6}$

so $z = 2e^{i\left(\pi/24 + \frac{2n\pi}{4}\right)} = 2e^{i\left(\pi/24 + \frac{n\pi}{2}\right)}$



$n=0 \quad z = 2e^{i\pi/24}$

$n=1 \quad z = 2e^{\frac{i13\pi}{24}}$

$n=-1 \quad z = 2e^{-\frac{i11\pi}{24}}$

$n=-2 \quad z = 2e^{-\frac{i23\pi}{24}}$

f) $z^6 = i = e^{i\pi/2}$

$z = e^{i\left(\pi/12 + \frac{2n\pi}{6}\right)} = e^{i\left(\pi/12 + \frac{n\pi}{3}\right)}$

$n=0 \quad z = e^{i\pi/12}$

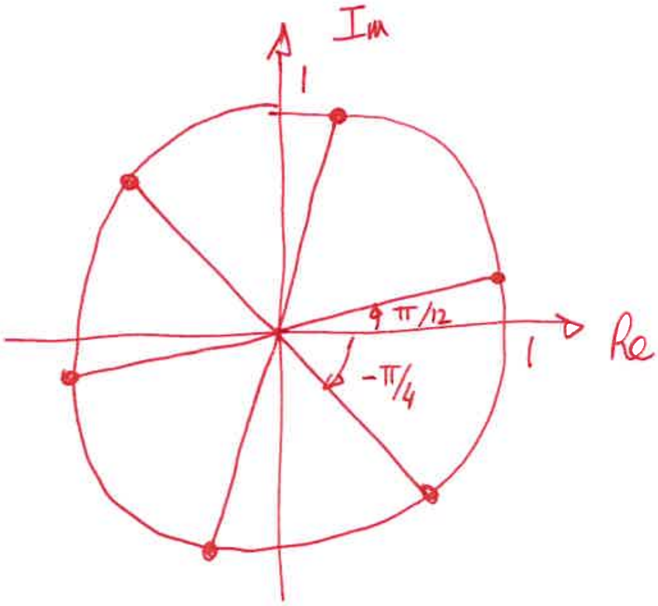
$n=1 \quad z = e^{\frac{i5\pi}{12}}$

$n=2 \quad z = e^{\frac{i3\pi}{4}}$

$n=-1 \quad z = e^{-\frac{i\pi}{4}}$

$n=-2 \quad z = e^{-\frac{i7\pi}{12}}$

$n=-3 \quad z = e^{-\frac{i11\pi}{12}}$

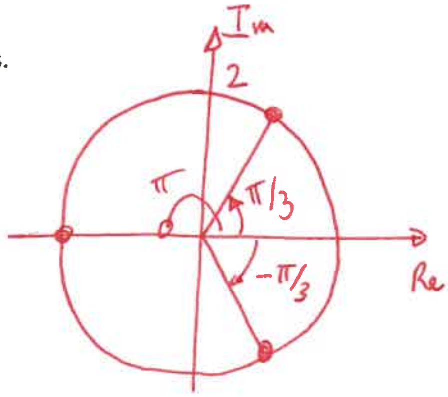


ROOTS OF COMPLEX NUMBERS

4 The point $1 + \sqrt{3}i$ and two other points are on the circumference of a circle with centre O and radius 2. The three points are the vertices of an equilateral triangle.

(a) Find the complex numbers represented by the two other points.

(b) Find the cubic equation that has these three complex numbers as its roots.



a) $1+\sqrt{3}\,i = 2e^{i\pi/3}$

We need to rotate this point by $\pm\frac{2\pi}{3}$

The 1st point is $2e^{i\pi/3}e^{i2\pi/3} = 2e^{i\pi} = -2$

the 2nd point is $2e^{i\pi/3}e^{-i2\pi/3} = 2e^{-i\pi/3}$

$\left[\text{or } 2e^{-i\pi/3} = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}\right]$

b) The cubic expression could be factorised as

$(z+2)\left[z-(1+\sqrt{3}i)\right]\left[z-(1-i\sqrt{3})\right] = (z+2)\left[z^2 - z(1+\sqrt{3}i+1-\sqrt{3}i) + (1+\sqrt{3}i)(1-i\sqrt{3})\right]$

$= (z+2)\left[z^2 - 2z + (1 - i\sqrt{3} + i\sqrt{3} + 3)\right]$

$= (z+2)\left[z^2 - 2z + 4\right]$

$= z^3 + z^2(2-2) + z(4-4) + 8$

$= z^3 + 8$

So the cubic equation that has these 3 roots is

$z^3 = -8$

ROOTS OF COMPLEX NUMBERS

$z^3 = 1$ so $z^3 - 1 = 0$

5 If 1, w_1 and w_2 are the cube roots of unity, prove the following:

(a) $w_1 = \overline{w_2} = w_2^2$ **(b)** $w_1 + w_2 = -1$ **(c)** $w_1 w_2 = 1$

a) $z^3 = 1$ so $z = e^{i\left(\frac{2n\pi}{3}\right)}$

$n = 0 \quad z = 1$

$n = 1 \quad z = e^{\frac{i2\pi}{3}} = w_1$

$n = -1 \quad z = e^{\frac{-i2\pi}{3}} = w_2$

indeed $w_2 = \overline{w_1}$

and $w_2^2 = \left(e^{\frac{-i2\pi}{3}}\right)^2 = e^{\frac{-i4\pi}{3}} = e^{\frac{i2\pi}{3}}$

so indeed $w_1 = \overline{w_2} = w_2^2$

b) $w_1 + w_2 + 1 = -\frac{b}{a}$ where $az^3 + bz^2 + cz + d = 0$

sum of roots of cubic

so $b = 0$

So $w_1 + w_2 = -1$

c) Product of roots of cubic equation

$w_1 w_2 \times 1 = -\frac{d}{a} = \frac{-(-1)}{1} = 1$ and $d = -1$

so indeed $w_1 w_2 = 1$

ROOTS OF COMPLEX NUMBERS

6 If w is a non-real cube root of unity (i.e. w is a non-real root of $z^3 = 1$), show the following:

(a) $1 + w + w^2 = 0$ **(b)** $(1 - w)(1 - w^2) = 3$

Now evaluate the following:

(c) $(1 + w)^3$ **(d)** $(1 + 2w + 3w^2)(1 + 2w^2 + 3w)$ **(e)** $(w^2 + 2w + w^3)(2w^2 + w + w^3)$

(f) $(1 - w)(1 - w^2)(1 - w^4)(1 - w^5)(1 - w^7)(1 - w^8)$

a) $w^3 = 1$ so $w^3 - 1 = 0$ or $(w-1)(w^2+w+1) = 0$

w is non real, so $\neq 1$, so $[w^2+w+1]$ must be zero.

b) $(1-w)(1-w^2) = 1 + w^3 - w^2 - w$

$= 1 + 1 - (w^2 + w)$

$= 2 - (-1)$ as $w^2 + w + 1 = 0$

$= 3$

c) $(1+w)^3 = 1 + 3w + 3w^2 + \underbrace{w^3}_{1} = 2 + 3(w+w^2) = 2 + 3(-1) = -1$

d) $(1+2w+3w^2)(1+2w^2+3w) = [1+2w+3(-w-1)][1+3w+2(-w-1)]$

______ $= [-2-w][-1+w] = -w^2 + w(1-2) + 2$

______ $= 1 + w - w + 2 = 3$

e) $[w^2+2w+w^3][2w^2+w+w^3] = w^2[w+2+w^2][2w+1+w^2]$

______ $= w^2[\cancel{w}+2-1-\cancel{w}][2w+1-w-1]$

______ $= w^2 \times 1 \times [w] = w^3 = 1$

f) Now we remember that $w^3 = 1$ so $w^4 = w$ and $w^5 = w^2$, etc

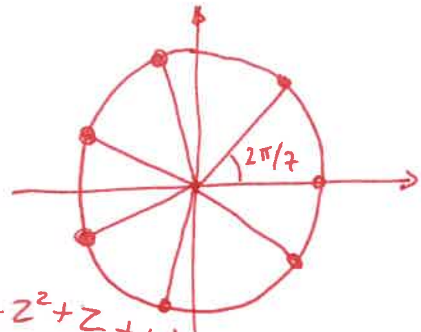
$\underbrace{(1-w)(1-w^2)}_{3}(1-w^4)(1-w^5)(1-w^7)(1-w^8) = 3 \times \underbrace{(1-w)(1-w^2)}_{=3}\underbrace{(1-w)(1-w^2)}_{=3}$

So $(1-w)(1-w^2)(1-w^4)(1-w^5)(1-w^7)(1-w^8) = 3^3 = 27$

ROOTS OF COMPLEX NUMBERS

8 **(a)** Find the roots of $z^7 = 1$ in mod–arg form and show them on an Argand diagram.
(b) If w is a non-real root, show that $w + w^2 + w^3 + w^4 + w^5 + w^6 = -1$.
(c) Show that the quadratic equation $z^2 + z + 2 = 0$ has roots $w + w^2 + w^4$ and $w^3 + w^5 + w^6$.
(d) Show that $\cos\frac{\pi}{7} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \frac{1}{2}$.

a) $z = e^{i\frac{2\pi}{7}n}$



b) $z^7 - 1 = 0$

But also $z^7 - 1 = (z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

So if z is a non real root, ~~we must have~~ the 2nd term must be 0. i.e. $z^6+z^5+z^4+z^3+z^2+z+1=0$

c) $z^2+z+2=0$

if $(w+w^2+w^4)$ and $(w^3+w^5+w^6)$ are roots of this equation

Then $w+w^2+w^4+w^3+w^5+w^6 = -\frac{b}{a} = -1$ which is true

and $(w+w^2+w^4)(w^3+w^5+w^6) = w^4+w^6+w^7+w^5+w^7+w^8+w^7+w^9+w^{10}$

$= w^4+w^6+1+w^5+1+\underbrace{w^7}_{=1}\times w + 1 + w^2\times w^7 + w^3\times w^7$

$= w^4+w^6+1+w^5+1+w+1+w^2+w^3$

$= \underbrace{w^6+w^5+w^4+w^3+w^2+w+\cancel{1}}_{=0} + 2$

$= 1$

The product of the roots must be equal to $\frac{c}{a}$ which is $\frac{2}{1}=2$

So indeed $(w+w^2+w^4)$ and $(w^3+w^5+w^6)$ must be roots of this equation.

ROOTS OF COMPLEX NUMBERS

11 **(a)** Show that $z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a root of $z^4 + z^3 + z^2 + z + 1 = 0$.

(b) Find all four roots of $z^4 + z^3 + z^2 + z + 1 = 0$.

(c) Show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

(d) Deduce that $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$.

a) $z_1 = e^{i2\pi/5}$ so $z_1^5 = 1$ or $z_1^5 - 1 = 0$

But $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$

So as z_1 is not a real root, it must be a root of the 2nd term, i.e. $z_1^4 + z_1^3 + z_1^2 + z_1 + 1 = 0$

b) From $z^5 = 1 = e^{i2n\pi}$ so $z = e^{i\frac{2n\pi}{5}}$

$n=0$ not that one as it's not a solution of $z^4+z^3+z^2+z+1=0$

$n=1$ $z = e^{i2\pi/5}$ $\quad n=-1$ $e^{\frac{-i2\pi}{5}}$

$n=2$ $z = e^{i4\pi/5}$ $\quad n=-2$ $z = e^{-i4\pi/5}$

c) Both $z = e^{i2\pi/5}$ and $z = e^{i4\pi/5}$ satisfy $z^4+z^3+z^2+z=-1$

$(e^{i2\pi/5})^4 + (e^{i2\pi/5})^3 + (e^{i2\pi/5})^2 + (e^{i2\pi/5}) = -1$

so $e^{i8\pi/5} + e^{i6\pi/5} + e^{i4\pi/5} + e^{i2\pi/5} = -1$

OR $e^{-i2\pi/5} + e^{-i4\pi/5} + e^{i4\pi/5} + e^{i2\pi/5} = -1$

$\left[\cos\left(\frac{-2\pi}{5}\right) + i\sin\left(\frac{-2\pi}{5}\right)\right] + \left[\cos\left(\frac{-4\pi}{5}\right) + i\sin\left(\frac{-4\pi}{5}\right)\right] + \left[\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right] + \left[\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right] = -1$

OR $\cos\left(\frac{2\pi}{5}\right) \cancel{- i\sin\left(\frac{2\pi}{5}\right)} + \cos\left(\frac{4\pi}{5}\right) \cancel{- i\sin\left(\frac{4\pi}{5}\right)} + \cos\left(\frac{4\pi}{5}\right) \cancel{+ i\sin\left(\frac{4\pi}{5}\right)} + \cos\left(\frac{2\pi}{5}\right) \cancel{+ i\sin\left(\frac{2\pi}{5}\right)} = -1$

$2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = -1$

so $\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$

OR FROM $e^{-i2\pi/5} + e^{-i4\pi/5} + e^{i4\pi/5} + e^{i2\pi/5} = -1$

We remember that $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ $\left[\text{and } \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}\right]$

So $e^{ix} + e^{-ix} = 2\cos x$ and therefore:

$$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$$

$$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$

d) We know that $\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$

But $\cos 2\theta = 2\cos^2\theta - 1$ so $\cos\left(\frac{4\pi}{5}\right) = 2\cos^2\left(\frac{2\pi}{5}\right) - 1$

$$\therefore \cos\left(\frac{2\pi}{5}\right) + 2\cos^2\left(\frac{2\pi}{5}\right) - 1 = -\frac{1}{2}$$

Change of variable $X = \cos\left(\frac{2\pi}{5}\right)$

$$X + 2X^2 - \frac{1}{2} = 0 \quad \text{OR} \quad 4X^2 + 2X - 1 = 0$$

$$\Delta = 4 - 4 \times (-1) \times 4 = 20 \qquad \sqrt{\Delta} = \sqrt{20} = 2\sqrt{5}$$

$$X = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

But $\cos\left(\frac{2\pi}{5}\right) > 0$ so we only keep $\dfrac{-1+\sqrt{5}}{4}$

$$\therefore \cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}$$