Question 1: Find the exact values

a) $arcsin\ 1 = \pi/2$	b) $arcsin\ 0 = 0$	c) $arcsin(-1) = -\frac{\pi}{2}$	d) $arcsin\left(\frac{\sqrt{3}}{2}\right) = \pi/3$
e) $arcsin\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\pi}{3}$	f) $arccos\ 1 = 0$	g) $arccos\left(\frac{1}{2}\right) = \pi/3$	h) $arctan\ 1 = \pi/4$
i) $arctan\sqrt{3} = \pi/3$	j) $arctan\left(\frac{1}{\sqrt{3}}\right) = \pi/6$	k) $cos^{-1}\left(cos\left(-\frac{\pi}{3}\right)\right) = cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$	l) $tan^{-1}\left(tan\frac{7\pi}{6}\right) = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Question 2: Evaluate

a) $cos\left(sin^{-1}\left(\frac{1}{2}\right)\right) =$

$$= cos\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2}$$

b) $cos\left(arctan\left(-\sqrt{3}\right)\right) =$

$$= cos\left(\frac{-\pi}{3}\right)$$

$$= \frac{1}{2}$$

c) $sin\left(2\ tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) =$

$$= sin\left(2 \times \frac{\pi}{6}\right)$$

$$= sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}$$

d) $cos\left(2\ cos^{-1}\left(\frac{5}{13}\right)\right) =$

(tip: use double angle formula for cosine, i.e. $cos\ 2\theta = 2\ cos^2\theta - 1$)

$$= 2\cos^2\left(cos^{-1}\left(\frac{5}{13}\right)\right) - 1$$

$$= 2\left[cos\left(cos^{-1}\left(\frac{5}{13}\right)\right)\right]^2 - 1$$

$$= 2\left[\frac{5}{13}\right]^2 - 1$$

$$= 2 \times \frac{25}{169} - 1 = -\frac{119}{169}$$

e) $sec\left[sin^{-1}\left(-\frac{1}{3}\right)\right] = E$
(tip: you will need to use $sin^2x + cos^2x = 1$)

$$\sec\theta = 1/\cos\theta = \frac{1}{\sqrt{1-\sin^2\theta}}$$

So $$E = \frac{1}{\sqrt{1-\sin^2(\sin^{-1}(-1/3))}}$$

$$E = \frac{1}{\sqrt{1-[\sin(\sin^{-1}(-1/3))]^2}}$$

$$E = \frac{1}{\sqrt{1-(1/3)^2}} = \frac{1}{\sqrt{1-1/9}}$$

$$E = \frac{1}{\sqrt{8/9}} = \frac{3}{\sqrt{8}}$$

f) $cosec\left[cos^{-1}\left(\frac{2}{3}\right)\right] = A$
(tip: you will need to use $sin^2x + cos^2x = 1$)

$$\mathrm{cosec}\,\theta = 1/\sin\theta = \frac{1}{\sqrt{1-\cos^2\theta}}$$

So $$A = \frac{1}{\sqrt{1-\cos^2(\cos^{-1}(2/3))}}$$

$$A = \frac{1}{\sqrt{1-[\cos(\cos^{-1}(2/3))]^2}}$$

$$A = \frac{1}{\sqrt{1-(2/3)^2}} = \frac{1}{\sqrt{1-4/9}}$$

$$A = \frac{1}{\sqrt{5/9}} = \frac{3}{\sqrt{5}}$$

Question 3: Find the exact values of:

a) $sin\left[sin^{-1}\left(\frac{3}{5}\right)\right] + sin\left[sin^{-1}\left(-\frac{3}{5}\right)\right] = A$

~~let θ = sin⁻¹(3/5)~~

we know that $\sin^{-1}$ is an odd function so $\sin^{-1}(-3/5) = -\sin^{-1}(3/5)$. So

$$A = \sin(\sin^{-1}(3/5)) + \sin(-\sin^{-1}(3/5))$$

$$A = \sin(\sin^{-1}(3/5)) - \sin(\sin^{-1}(3/5)) = 0$$

b) $cos\left[2\,cos^{-1}\left(\frac{1}{3}\right)\right]$

$$= 2\cos^2\left[\cos^{-1}(1/3)\right] - 1$$

$$= 2\left[\cos(\cos^{-1}(1/3))\right]^2 - 1$$

$$= 2 \times \left(\frac{1}{3}\right)^2 - 1$$

$$= 2 \times \frac{1}{9} - 1 = \frac{-7}{9}$$

Question 4: Show that $f(x) = tan\,(cos^{-1}x)$ is an odd function

$f(-x) = \tan(\cos^{-1}(-x))$ But we know that the function $\cos^{-1}$ is symmetrical with regard to the point $(0, \frac{\pi}{2})$, so ~~tan~~

$$\cos^{-1}(-x) + \cos^{-1}(x) = \pi$$

So $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

So $f(-x) = \tan(\pi - \cos^{-1}(x))$ But the function tangent is periodic of period π, so

$f(-x) = \tan(-\cos^{-1}(x))$ But $\tan(-\theta) = -\tan\theta$, so

$f(-x) = -\tan(\cos^{-1}(x)) = -f(x)$ So f is an odd function

Question 5: Show that:

a) $\tan^{-1}(4) - \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

We use the equality $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

We calculate the tangent of the Left side.

$$\tan\left[\tan^{-1}(4) - \tan^{-1}(3/5)\right] = \frac{\tan\left(\tan^{-1}(4)\right) - \tan\left[\tan^{-1}(3/5)\right]}{1 + \tan\left[\tan^{-1}(4)\right]\tan\left[\tan^{-1}(3/5)\right]} = \frac{4 - 3/5}{1 + 4 \times 3/5} = 1$$

But $1 = \tan\frac{\pi}{4}$

So $\tan\left[\tan^{-1}(4) - \tan^{-1}(3/5)\right] = 1 = \tan\left[\frac{\pi}{4}\right]$

So we must have $\tan^{-1}(4) - \tan^{-1}(3/5) = \pi/4$

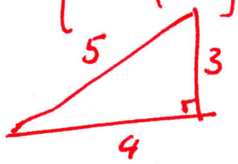
b) $\cos^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(-\frac{3}{4}\right) = \frac{\pi}{2}$

$0 < \cos^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{2}$ and $0 < \tan^{-1}\left(\frac{3}{4}\right) < \pi/2$

So $0 < \cos^{-1}(3/5) + \tan^{-1}(3/4) < \pi$

$$\cos\left[\cos^{-1}(3/5) - \tan^{-1}(-3/4)\right] = \cos\left[\cos^{-1}(3/5) + \tan^{-1}(3/4)\right]$$

$$= \underbrace{\cos\left[\cos^{-1}(3/5)\right]}_{3/5}\ \underbrace{\cos\left[\tan^{-1}(3/4)\right]}_{=4/5} - \underbrace{\sin\left[\cos^{-1}(3/5)\right]}_{4/5}\ \overbrace{\sin\left[\tan^{-1}(3/4)\right]}^{3/5}$$



$$= \frac{3}{5} \times \frac{4}{5} - \frac{4}{5} \times \frac{3}{5} = 0 = \cos\left(\frac{\pi}{2}\right)$$

So we must have $\cos^{-1}(3/5) - \tan^{-1}(-3/4) = \frac{\pi}{2}$

c) $\sin^{-1}\left(\frac{5}{13}\right) + \tan^{-1}\left(\frac{16}{63}\right) = \cos^{-1}\left(\frac{4}{5}\right)$

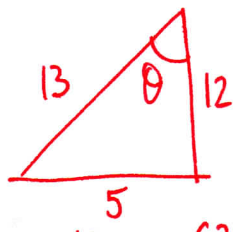
$0 < \sin^{-1}(5/13) < \frac{\pi}{2}$ and $0 < \tan^{-1}\left(\frac{16}{63}\right) < \frac{\pi}{2}$

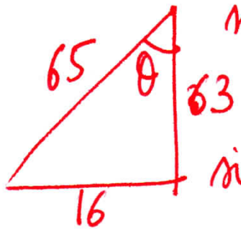
So $0 < \sin^{-1}\left(\frac{5}{13}\right) + \tan^{-1}\left(\frac{16}{63}\right) < \pi$

We take the cosine of the left hand side.

$$(A) = \cos\left[\sin^{-1}\left(\frac{5}{13}\right) + \tan^{-1}\left(\frac{16}{63}\right)\right] = \cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right]\cos\left[\tan^{-1}\left(\frac{16}{63}\right)\right] - \sin\left[\sin^{-1}\left(\frac{5}{13}\right)\right]\sin\left[\tan^{-1}\left(\frac{16}{63}\right)\right]$$

so $\cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right] = \frac{12}{13}$





so $\cos\left[\tan^{-1}\left(\frac{16}{63}\right)\right] = \frac{63}{65}$

$\sin\left[\tan^{-1}\left(\frac{16}{63}\right)\right] = \frac{16}{65}$

$$(A) = \frac{12}{13} \times \frac{63}{65} - \frac{5}{13} \times \frac{16}{65} = \frac{676}{845} = \frac{4}{5}$$

$\therefore \sin^{-1}\left(\frac{5}{13}\right) + \tan^{-1}\left(\frac{16}{63}\right) = \cos^{-1}\left(\frac{4}{5}\right)$

Question 6: Find the exact values of:

c) $\cos\left[\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right] = E$

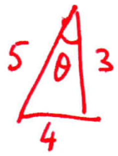
(tip: use the formula

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and also

$\sin^2 x + \cos^2 x = 1$)

$$E = \cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right]\cos\left[\sin^{-1}\left(\frac{4}{5}\right)\right] - \sin\left[\sin^{-1}\left(\frac{5}{13}\right)\right]\sin\left[\sin^{-1}\left(\frac{4}{5}\right)\right]$$

so $\cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right] = \frac{12}{13}$



so $\cos\left[\sin^{-1}\left(\frac{4}{5}\right)\right] = \frac{3}{5}$

$$\therefore E = \frac{12}{13} \times \frac{3}{5} - \frac{5}{13} \times \frac{4}{5}$$

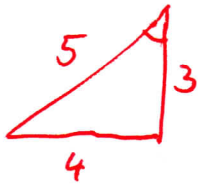
$$E = \frac{16}{65}$$

d) $\sin\left[2\tan^{-1}\left(\frac{4}{3}\right)\right] = A$

(tip: use double angle formula for sine, i.e.

$\sin 2\theta = 2\sin\theta\cos\theta$

$$A = 2\sin\left[\tan^{-1}\left(\frac{4}{3}\right)\right]\cos\left[\tan^{-1}\left(\frac{4}{3}\right)\right]$$



so $\sin\left[\tan^{-1}\left(\frac{4}{3}\right)\right] = \frac{4}{5}$

and $\cos\left[\tan^{-1}\left(\frac{4}{3}\right)\right] = \frac{3}{5}$

$$\therefore A = 2 \times \frac{4}{5} \times \frac{3}{5} = \frac{24}{25}$$

Question 6: If $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$, determine the inverse function f^{-1} and specify its domain and range.

$y = 3\cos^{-1}\left(\frac{x}{2}\right)$ To find the inverse function, we substitute x and y.

$$x = 3\cos^{-1}\left(\frac{y}{2}\right) \iff \frac{x}{3} = \cos^{-1}(y/2) \text{ so } \cos\left(\frac{x}{3}\right) = y/2$$

$$\therefore y = 2\cos\left(\frac{x}{3}\right) \qquad f^{-1}(x) = 2\cos\left(\frac{x}{3}\right)$$

Domain of f^{-1} : all values of $\mathbb{R}$ (all real values)

Range $[-2, 2]$.

Question 7: Solve simultaneously the two equations $2\sin^{-1}x + \cos^{-1}y = -\frac{\pi}{12}$ and $\sin^{-1}x - 2\cos^{-1}y = -\frac{2\pi}{3}$

Let $X = \sin^{-1}x$ and $Y = \cos^{-1}y$. The 2 equations become:

$2X + Y = -\pi/12$ and $X - 2Y = -\frac{2\pi}{3}$ ②

(or $4X + 2Y = -\pi/6$) ① By adding ① and ② (elimination method) we get

$5X = -\frac{2\pi}{3} - \frac{\pi}{6} = -\frac{5\pi}{6}$ so $\boxed{X = -\pi/6}$ and $Y = \frac{-\pi}{12} - 2\times\left(\frac{-\pi}{6}\right) = \frac{\pi}{4}$

$\boxed{Y = \pi/4}$

So $\sin^{-1}x = -\frac{\pi}{6}$ so $\boxed{x = -\frac{1}{2}}$

and $\cos^{-1}y = \frac{\pi}{4}$ so $\boxed{y = \frac{\sqrt{2}}{2}}$

Question 8:

a) Let $\theta = \sin^{-1}x$. Use the fact that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ to show that $\cos^{-1}x = \frac{\pi}{2} - \theta$

$\theta = \sin^{-1}x$ so $\sin\theta = x$

But $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ so $\cos\left(\frac{\pi}{2} - \theta\right) = x$

$\therefore \cos^{-1}\left[\cos\left(\frac{\pi}{2} - \theta\right)\right] = \cos^{-1}x$

OR $\frac{\pi}{2} - \theta = \cos^{-1}x$ Equation ①

b) deduce that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

From ①, as $\theta = \sin^{-1}x$, we get $\frac{\pi}{2} - \sin^{-1}x = \cos^{-1}x$

or $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$