FURTHER APPLICATIONS OF SERIES

1 Noor invests \$10 000 on the condition that the money is repaid in 12 equal quarterly instalments. If the investment earns interest at the rate of 1% per quarter, what is the amount of each instalment? Q

Amount owed after 1st repayment: $10{,}000 \times 1.01 - Q$

Amount owed after 2nd repayment: $[10{,}000 \times 1.01 - Q] \times 1.01 - Q$

$$= [10{,}000 \times 1.01^2 - Q(1 + 1.01)]$$

——— 3rd repayment: $[10{,}000 \times 1.01^2 - Q(1 + 1.01)] \times 1.01 - Q$

$$= [10{,}000 \times 1.01^3 - Q(1 + 1.01 + 1.01^2)]$$

$\vdots$

——— 12th repayment:

$$= [10{,}000 \times 1.01^{12} - Q(1 + 1.01 + 1.01^2 + \dots + 1.01^{11})]$$

and this amount is equal to zero (no more payments).

So $10{,}000 \times 1.01^{12} - Q(1 + 1.01 + 1.01^2 + \dots + 1.01^{11}) = 0$

$$\iff 10{,}000 \times 1.01^{12} - Q\left(\frac{1.01^{12} - 1}{1.01 - 1}\right) = 0$$

$$\left(\text{as } 1 + r + r^2 + \dots + r^{n-1} = \frac{r^n - 1}{r - 1} \text{ when } r > 1\right)$$

$$\iff 10{,}000 \times 1.01^{12} = Q\left(\frac{1.01^{12} - 1}{0.01}\right)$$

$$\therefore Q = \frac{10{,}000 \times 1.01^{12} \times 0.01}{(1.01^{12} - 1)} = 888.49$$

FURTHER APPLICATIONS OF SERIES

2 Ava invests \$20 000 on the condition that she is repaid the money in 16 equal quarterly instalments. If the investment earns interest at the rate of 0.75% per quarter, what is the amount of each instalment? Q

Amount owed after 1st repayment : $20{,}000 \times 1.0075 - Q$

——— 2nd ——— : $(20{,}000 \times 1.0075 - Q) \times 1.0075 - Q$

$= 20{,}000 \times 1.0075^2 - Q(1 + 1.0075)$

——— 3rd ——— : $[20{,}000 \times 1.0075^2 - Q(1 + 1.0075)] \times 1.0075 - Q$

——— $= 20{,}000 \times 1.0075^3 - Q(1 + 1.0075 + 1.0075^2)$

⋮

——— 16th ——— :

$= 20{,}000 \times 1.0075^{16} - Q(1 + 1.0075 + 1.0075^2 + \dots + 1.0075^{1}$

which is equal to zero (no more payments).

So $20{,}000 \times 1.0075^{16} = Q(1 + 1.0075 + 1.0075^2 + \dots + 1.0075^{15})$

——— $= Q\left[\dfrac{1.0075^{16} - 1}{1.0075 - 1}\right]$

So $Q = \dfrac{20{,}000 \times 1.0075^{16} \times (1.0075 - 1)}{(1.0075^{16} - 1)}$

$Q = 1{,}331.18$

FURTHER APPLICATIONS OF SERIES

3 **(a)** When Nazeera started a new job, \$400 was deposited into her superannuation fund at the beginning of each month. The money was invested at 0.4% per month, compounded monthly.
Let \$$Y$ be the value of the investment after 360 months, when Nazeera retires. Show that $Y = 322\,142.43$.

(b) After retirement, Nazeera withdraws \$3000 from her superannuation fund at the end of each month without making any further deposits. The account continues to earn interest at 0.4% per month.
Let \$$A_n$ be the amount of money left in the account n months after Nazeera's retirement.

(i) Show that $A_n = (Y - 750\,000) \times 1.004^n + 750\,000$.

(ii) For how many months after retirement will there be money left in the account?

a) First payment earns 400×1.004^{360} (360 periods)

2nd payment earns 400×1.004^{359} (359 periods)

⋮

last payment earns 400×1.004

So in total $Y = 400\left(1.004 + \dots + 1.004^{360}\right)$

$$Y = 400 \times 1.004 \times \left(1 + 1.004 + \dots + 1.004^{359}\right)$$

$$Y = 400 \times 1.004 \times \left(\frac{1.004^{360} - 1}{1.004 - 1}\right) = 322,142.43$$

b) i) $A_1 = Y \times 1.004 - 3,000$

$$A_2 = [Y \times 1.004 - 3,000] \times 1.004 - 3,000 = Y \times 1.004^2 - 3,000(1 + 1.004$$

$$A_3 = [Y \times 1.004^2 - 3,000(1 + 1.004)] \times 1.004 - 3000$$

$$= Y \times 1.004^3 - 3,000\left(1 + 1.004 + 1.004^2\right)$$

⋮

$$A_n = Y \times 1.004^n - 3,000 \times \left(\frac{1.004^n - 1}{0.004}\right)$$

$$A_n = (Y - 750,000) \times 1.004^n + 750,000$$

ii) $A_n = 0$ when $(Y - 750,000) \times 1.004^n = -750,000$

i.e : $1.004^n = \dfrac{750,000}{750,000 - Y} = \dfrac{750,000}{750,000 - 322,142.43} = 1.75292$

$n = \dfrac{\ln(1.75292)}{\ln(1.004)} = 141$ months so approx 11.75 years.

so 0.5%/month

4 One year ago William and Kate borrowed \$400 000 to buy an apartment. The interest rate was 6% p.a., compounded monthly. They agreed to repay the loan over 25 years with equal monthly repayments of \$2578.

(a) Calculate how much money William and Kate owed after their first monthly repayment.

(b) After making their twelfth monthly repayment, William and Kate owe \$392 870. The interest rate now increases to 9% p.a., compounded monthly. The amount $\$A_n$ owing on the loan after the nth monthly repayment is now calculated using the formula $A_n = 392\,870 \times 1.0075^n - 1.0075^{n-1}M - \ldots - 1.0075M - M$, where $\$M$ is the monthly repayment and $n = 1, 2, \ldots 288$. (Do **not** prove this formula.)
Calculate the monthly repayment if the loan is to be repaid over the remaining 24 years (288 months).

(c) William and Kate now decide to increase their monthly repayments to \$3500. How long will it take them to repay \$392 870?

(d) How much money will William and Kate save over the term of the loan by making these higher monthly repayments, compared to keeping the repayments at the amount calculated in **(b)**?

a) Amount owed after 1st repayment : $400{,}000 \times 1.005 - 2578 = 399{,}422$

b) $A_n = 392\,870 \times 1.0075^n - 1.0075^{n-1}M - \ldots - 1.005M - M$

$A_n = 392\,870 \times 1.0075^n - M(1 + 1.0075 + \ldots + 1.0075^{n-1})$

$A_n = 392\,870 \times 1.0075^n - M\left(\frac{1.0075^n - 1}{0.0075}\right)$

We know that $A_{288} = 0$, so

$M\left(\frac{1.0075^{288} - 1}{0.0075}\right) = 392{,}870 \times 1.0075^{288}$

$M = \frac{392{,}870 \times 1.0075^{288} \times 0.0075}{1.0075^{288} - 1} = 3{,}334.15$

c) we look for n such that: $A_n = 0$

$0 = 392{,}870 \times 1.0075^n - 3500 \times \left(\frac{1.0075^n - 1}{0.0075}\right)$

$1.0075^n\left(392{,}870 - \frac{3{,}500}{0.0075}\right) = -\frac{3500}{0.0075}$

$1.0075^n(-553.475) = -3500$ so $1.0075^n = \frac{3500}{553.475} = 6.32368$

$n = \frac{\ln(6.32368)}{\ln(1.0075)} = 247$ months

d) Amount saved $= 288 \times 3{,}334.15 - 247 \times 3500$

$\simeq 95{,}735$

5 Which will give the better financial result after 20 years: a lump sum of $100 000 invested at 5% p.a. compounded annually, or a monthly payment of $600 with interest at 5% p.a. compounded monthly?

$5/12 =$

Case 1

$$P = 100{,}000 \times (1 + 0.05)^{20} = 265{,}329.77$$

Case 2 Amount after 1 month : $600 \times \left(1 + \frac{0.05}{12}\right) = 600\left(1 + \frac{1}{240}\right)$

Amount after 2 months : $\left[600\left(1 + \frac{1}{240}\right)\right] \times \left(1 + \frac{1}{240}\right) + 600\left(1 + \frac{1}{240}\right)$

$$= 600\left(1 + \frac{1}{240}\right)^2 + 600\left(1 + \frac{1}{240}\right)$$

Amount after 3 months : $\left[600\left(1 + \frac{1}{240}\right)^2 + 600\left(1 + \frac{1}{240}\right)\right] \times \left(1 + \frac{1}{240}\right) + 600\left(1 + \frac{1}{240}\right)$

$$= 600\left[1 + \left(1 + \frac{1}{240}\right) + \left(1 + \frac{1}{240}\right)^2\right]$$

$\vdots$

After 240 months: $600\left[1 + \left(1 + \frac{1}{240}\right) + \left(1 + \frac{1}{240}\right)^2 + \dots + \left(1 + \frac{1}{240}\right)^{599}\right]$

So equal to $600\left[\frac{\left(\frac{241}{240}\right)^{240} - 1}{\frac{241}{240} - 1}\right] = 246{,}620.20$

So the lump sum investment (Case 1) is best

FURTHER APPLICATIONS OF SERIES

6 A lottery offers a prize of \$100 000 immediately, or \$10 000 now plus \$10 000 per year for the next 11 years. You take the prize of \$100 000, keep \$10 000 to spend over the next year and invest the remaining \$90 000 as an annuity at 5% p.a. You plan to withdraw \$10 000 at the end of each year for the next 11 years.

(a) Is this a realistic plan?
(b) How much money is left in your annuity after the eleventh payment?
(c) How much money is left in your annuity after the twelfth payment?

a) Money left at the end of 1st year: $90{,}000 \times 1.05 - 10{,}000$

2nd —: $[90{,}000 \times 1.05 - 10{,}000] \times 1.05 - 10{,}000$

$= 90{,}000 \times 1.05^2 - 10{,}000\,[1 + 1.05]$

⋮

n years: $A_n = 90{,}000 \times 1.05^n - 10{,}000\,[1 + 1.05 + \dots + 1.05^{n-1}]$

$$A_n = 90{,}000 \times 1.05^n - 10{,}000 \left[\frac{1.05^n - 1}{0.05}\right]$$

$$A_n = 90{,}000 \times 1.05^n - 200{,}000\,(1.05^n - 1)$$

$$A_n = -110{,}000 \times 1.05^n + 200{,}000$$

when $n = 10$ $A_{10} = 20{,}821.59$

— $n = 11$ $A_{11} = 11{,}862.67$ so there's still money left, it's a realistic plan.

b) $11{,}862.67$ as calculated above

c) $n = 12$ $A_{12} = 200{,}000 - 110{,}000 \times 1.05^{12}$

$A_{12} = 2455.80$

, so 0.00225/m

7 Prior to retirement, Cassie deposits $100 000 in an account which will earn interest at a rate of 2.7% p.a., paid monthly. Cassie wishes to withdraw $300 per month.

(a) Set up a recurrence relation that models this account and gives the balance after n payments.
(b) What is the balance after two withdrawals?
(c) Explain what will happen to the balance if the interest was 5% per annum from the start.
(d) What monthly payment would keep the balance at $100 000? Calculate the answer for rates of 5% and 2.7%.

a) Balance after 1 payment: $100{,}000 \times 1.00225 - 300$

Balance after 2 payments: $\underbrace{[100{,}000 \times 1.00225 - 300] \times 1.00225 - 300}_{= A_2}$

——— 3 payments: $A_3 = A_2 \times 1.00225 - 300.$

So the recurrence relation is $A_n = A_{n-1} \times 1.00225 - 300$

b) $A_1 = 100{,}000 \times 1.00225 - 300 = 99{,}925$

$A_2 = 99925 \times 1.00225 - 300 = 99{,}849.83$

c) The recurrence relation would be instead

$$A_n = A_{n-1} \times \left(1 + \frac{0.05}{12}\right) - 300 = A_{n-1} \times \frac{241}{240} - 300$$

$$A_1 = 100{,}000 \times \frac{241}{240} - 300 = 100{,}116.6\dot{6}$$

So the balance would increase, instead of decreasing

d) If the rate was 2.7%

$$A_n = A_{n-1} \times 1.00225 - M$$

For A_n to be equal to A_{n-1}, and to 100,000, the monthly payment should be

$$M = (1.00225 - 1) \times 100{,}000$$

$$M = 225$$

If the rate was 5%

Then $A_n = A_{n-1} \times \frac{241}{240} - M$

For $A_n = A_{n-1} = 100,000$, we would have

$$M = \left(\frac{241}{240} - 1\right) \times 100,000$$

$$M = 416.67$$