METHODS OF PROOF

1 Use a direct proof to prove each of the following statements.

(a) The sum of any two odd integers is even.
(b) The sum of an odd integer and an even integer is always odd.
(c) The product of two odd integers is odd.
(d) The sum of two consecutive odd numbers is divisible by 4.
(e) The sum of the squares of five consecutive integers is divisible by 5.
(f) The product of two rational numbers is rational.
(g) The sum of two rational numbers is rational.
(h) If n is odd, then n^2 is odd.

a) $n = 2k+1$ and $m = 2\ell+1$ so $n+m = 2k+1+2\ell+1 = 2(k+\ell)+2 = 2(k+\ell+1)$

$\therefore$ so $(n+m)$ is even.

b) $\exists k \in \mathbb{Z}$ and $\exists \ell \in \mathbb{Z}$ such that $n = 2k+1$ and $m = 2\ell$.

So $n+m = 2k+1+2\ell = 2(k+\ell)+1$ so $(n+m)$ is odd.

c) let $n = 2k+1$ and $m = 2\ell+1$

so $nm = (2k+1)(2\ell+1) = 2k\ell + 2k + 2\ell + 1 = 2[k\ell+k+\ell]+1$

$\therefore$ nm is odd

d) let $n = 2k+1$ and $n+2 = 2k+3$

So $n+m = 2k+1+2k+3 = 4k+4 = 4(k+1)$

$\therefore$ $(n+m)$ is divisible by 4

e) $n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2 = 5n^2+2n+4n+6n+8n+30$

$= 5n^2+20n+30 = 5[n^2+4n+6]$ $\therefore$ divisible by 5

f) let $x = \frac{p}{q}$ and $y = \frac{n}{m}$ $\therefore$ $xy = \frac{pn}{qm}$ rational.

g) let $x = \frac{p}{q}$ and $y = \frac{n}{m}$ $\therefore$ $x+y = \frac{p}{q}+\frac{n}{m} = \frac{pm+nq}{qm}$ which is rational.

h) $\exists k \in \mathbb{Z}$ such that $n = 2k+1$

$n^2 = (2k+1)^2 = 4k^2+4k+1 = \underbrace{4(k^2+k)}_{\text{even}}+1$

So n^2 is odd.

2 Use a contrapositive proof to prove each of the following statements.

(a) Let n be an integer. If $3n+2$ is even, then n is even.
(b) If a and b are integers and ab is even, then at least one of a and b is even.
(c) Let n be an integer. If n^3+5 is odd, then n is even.
(d) If x is irrational, then $\sqrt{x}$ is irrational.

a) The contrapositive statement is: if n is odd, then $(3n+2)$ is odd.
if n is odd, then $\exists\, k \in \mathbb{Z}$ such that $n = 2k+1$
So $(3n+2) = 3(2k+1)+2 = \underbrace{6k}_{\text{even}} + 5$ so odd.

$\therefore$ $(3n+2)$ is odd.
$\therefore$ if $(3n+2)$ is even, then n is even.

b) The contrapositive statement is: if neither a nor b is even, then ab is odd. (or if both a and b are odd, then ab is odd)
indeed, a odd $\Leftrightarrow \exists\, k \in \mathbb{Z}$ such that $a = 2k+1$ (and likewise $b = 2n+1$)
so $ab = (2k+1)(2n+1) = \underbrace{4nk + 2k + 2n}_{\text{even}} + 1$ so ab odd.

c) The contrapositive statement is: if n is odd, then (n^3+5) is even.
n odd $\Leftrightarrow \exists\, k \in \mathbb{Z}$ such that $n = 2k+1$
$n^3+5 = (2k+1)^3 + 5 = 8k^3 + 3 \times 4k^2 + 3 \times 2k + 1 + 5 = \underbrace{8k^3+12k^2+6k}_{\text{even}}+6$
So (n^3+5) is even. The contrapositive statement is true, so the original is true.

d) The contrapositive statement is: if $\sqrt{x}$ is rational, then x is rational
For $\sqrt{x}$ to be rational, $\exists\, p, q \in \mathbb{Z}$ such that $\sqrt{x} = \frac{p}{q}$
Then $x = \frac{p^2}{q^2}$ so x is rational.
The contrapositive statement is true, therefore the original statement must be true.

3 Use a proof by contradiction to prove each of the following statements.

(a) $\sqrt{3}$ is irrational.

~~**(b)** $\sqrt{5}$ is irrational~~

(c) The sum of a rational and an irrational number is irrational.

(d) The product of a rational and an irrational number is irrational.

* with p/q a fraction in its simplest form

a) Assume $\sqrt{3}$ is rational, i.e. $\exists\, p, q \in \mathbb{Z}$ such that $\sqrt{3} = \frac{p}{q}$ *

So $3 = \frac{p^2}{q^2}$ or $p^2 = 3q^2$ so p^2 is divisible by 3

So p must be divisible by 3, i.e. $\exists\, k \in \mathbb{Z}$ such that $p = 3k$.

So $3 = \frac{(3k)^2}{q^2} = \frac{9k^2}{q^2}$ So $q^2 = 3k^2$ so q^2 divisible by 3

so q divisible by 3. So $\frac{p}{q}$ is not a fraction in its simplest form.

So $\sqrt{3}$ is not rational.

c) Assume ~~that the sum~~ of a rational and an irrational number is a rational number, i.e $r = \frac{p}{q}$ x the irrational number.

So $\exists\, n, m \in \mathbb{Z}$ such that $r + x = \frac{m}{n}$ but $r = \frac{p}{q}$

So $x = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}$ so x is rational. which is a contradiction.

d) Assume that the product of a rational number $r = \frac{p}{q}$ and of an irrational number x is rational.

$\therefore\ \exists\, n, m \in \mathbb{Z}$ such that $r \times x = \frac{n}{m}$

or $x = \frac{n}{m} \times \frac{1}{r} = \frac{n}{m} \times \frac{q}{p} = \frac{nq}{mp}$

$\therefore$ x is rational. which is contradictory with the initial hypothesis.

4 Prove each of the following logical equivalences.

(a) Let n be a positive integer. $n+9$ is even if and only if $n+6$ is odd.
(b) Let n be a positive integer. $n-3$ is odd if and only if $n+2$ is even.
(c) Let n be a positive integer. n is even if and only if $13n+4$ is even.

a) $n+9$ even $\Rightarrow$ n must be odd.

$\Rightarrow$ $n+6$ must be odd.

Conversely

$n+6$ odd $\Rightarrow$ n must be odd

$\Rightarrow$ $n+9$ must be even.

So $(n+9)$ even $\Leftrightarrow$ $(n+6)$ odd

b) $n-3$ odd $\Rightarrow$ n must be even $\Rightarrow (n+2)$ even

Conversely $(n+2)$ even $\Rightarrow$ n must be even $\Rightarrow (n-3)$ is odd.

$\therefore$ $(n-3)$ odd $\Leftrightarrow$ $(n+2)$ even

c) n even $\Rightarrow \exists\, k \in \mathbb{Z}$ such that $n = 2k$.

$\Rightarrow 13n+4 = 13 \times 2k + 4 = 26k+4 = 2\underbrace{(13k+2)}_{\text{even}}$

$\Rightarrow$ So $(13n+4)$ is even.

For the converse:

$(13n+4)$ even $\Rightarrow \exists\, k \in \mathbb{Z}$ such that $13n+4 = 2k$

$\Rightarrow 13n = 2k-4 = 2(k-2)$.

$\Rightarrow$ So $13n$ is even

$\Rightarrow$ n must be even.

So n even $\Leftrightarrow$ $(13n+4)$ even

5 Consider the following statement:

'If two integers have an even product, then at least one of the two integers must be even.'

To prove this statement by contraposition, it would be necessary to:

A suppose that at least one of the two integers is even, and then show that the product must be even.
B suppose that at least one of the integers is odd, and then show that the product must be odd.
$\boxed{\textbf{C}}$ suppose that both integers are odd, and then show that the product must be odd.
D suppose that the two integers have an even product and that both integers are odd, and then show that a contradiction arises.

7 Let a, b, c be positive real numbers such that $ab = c$. Prove that $a \le \sqrt{c}$ or $b \le \sqrt{c}$.

Suppose, for contradiction, that both a and b are greater than $\sqrt{c}$.
i.e. $a > \sqrt{c}$ and $b > \sqrt{c}$
Then $ab > \sqrt{c} \times \sqrt{c}$ i.e. $ab > c$ which is a contradiction.
as $ab = c$
$\therefore$ it must be the case that at least one of a or b
must be less than $\sqrt{c}$.

9 Prove that a four-digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

let a, b, c, d the digits of a four digit number.
"abcd" $= 1000 \times a + 100 \times b + 10 \times c + d$
$\exists k \in \mathbb{Z}$ such that "abcd" $= 9k$
So $1000a + 100b + 10c + d = 9k$
$\Leftrightarrow \; a + b + c + d = 9k - 999a - 99b - 9c$
$\Leftrightarrow \; a + b + c + d = 9[k - 111a - 11b - c]$
So the sum of the digits must be divisible by 9.
For the converse: if $a + b + c + d = 9k$ with $k \in \mathbb{Z}$
Then the number "abcd" which is equal to $1000a + 100b + 10c + d$
can be rewritten as $999a + 99b + 9c + \underbrace{a+b+c+d}_{9k}$
So "abcd" can be rewritten as $999a + 99b + 9c + 9k$
which is equal to $9(111a + 11b + c + k)$ and therefore divisible by 9.

11 Prove that every odd integer can be expressed as the difference between two perfect squares.

If n is odd, then $\exists\, k \in \mathbb{Z}$ such that $n = 2k+1$

$n = 2k+1 = k^2 + 2k + 1 - k^2 = (k+1)^2 - k^2.$

So every odd integer can be expressed as the difference between 2 perfect squares.

by contradiction

12 Prove that if a, b are integers, then $a^2 - 4b - 3 \neq 0$.

By contradiction. Assume $\exists\, a, b \in \mathbb{Z}$ such that $a^2 - 4b - 3 = 0$.

So $a^2 = 4b + 3$ so a^2 must be odd. and so a must be odd.

i.e. $\exists\, k \in \mathbb{Z}$ such that $a = 2k+1$

So $(2k+1)^2 = 4b + 3 \iff 4k^2 + 4k + 1 = 4b + 3$

$\iff 4k^2 + 4k - 4b = 2$

$\iff k^2 + k - b = \frac{1}{2}$

The LHS of this equation is an integer, whereas the RHS is not, which is impossible.

Therefore there cannot be two integers a and b such that $a^2 - 4b - 3 = 0$

13 Let k be a positive integer. Prove that if $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+3} + 3^{3k+3}$ is also divisible by 5.

if $2^{k+2} + 3^{3k}$ is divisible by 5, there must exists an integer n such that $2^{k+2} + 3^{3k} = 5n$

which is equivalent to $2^{k+2} = 5n - 3^{3k}$

and thus $2^{k+3} = 2[5n - 3^{3k}]$

$$2^{k+3} = 10n - 2 \times 3^{3k}$$

Therefore $2^{k+3} + 3^{3k+3} = (10n - 2 \times 3^{3k}) + 3^{3k+3}$

$$= 10n - 2 \times 3^{3k} + 3^3 \times 3^{3k}$$

$$= 10n - 2 \times 3^{3k} + 27 \times 3^{3k}$$

$$= 10n + 3^{k}[-2 + 27]$$

$$= 10n + 25 \times 3^{k}$$

$$= 5(2n + 5 \times 3^{k})$$

Therefore the expession $(2^{k+3} + 3^{3k+3})$ is divisible by 5

15 Use a proof by contradiction to show that there is no rational solution to the equation $x^3 + x + 1 = 0$. As a hint, start by supposing, for a contradiction that $r = \frac{p}{q}$ is a rational solution to the equation, where p, q are integers with no common factor other than 1 and with $q \neq 0$. Then consider what would happen if both p and q were odd, or if one of them was even and the other odd.

Assume that $r = \frac{p}{q}$ is a rational solution to the equation, with p and q integers with no common factor other than 1.

1) If p and q are odd, i.e. $\exists\, n, m \in \mathbb{Z}$ such that $p = 2n+1$, $q = 2m+1$

then $x^3 + x = \frac{p^3}{q^3} + \frac{p}{q} = \frac{p^3 + pq^2}{q^3}$

$$x^3 + x = \frac{(2n+1)^3 + (2n+1)(2m+1)^2}{(2m+1)^3}$$

But $(2n+1)^3$ is odd and $(2n+1)(2m+1)^2$ is also odd.

therefore the numerator is even.

But the denominator is odd, so $(x^3 + x)$ is a fraction, and cannot be equal to (-1).

2) if p even and q odd then $p = 2n$ and $q = 2m+1$

then $x^3 + x = \left[\frac{2n}{(2m+1)}\right]^3 + \frac{2n}{2m+1}$

$$x^3 + x = \frac{(2n)^3(2m+1)^2 + 2n}{(2m+1)^3} = \frac{2n\left(4n^2(2m+1)+1\right)}{(2m+1)^3}$$

So the numerator is even, but the denominator is odd.

So x^3+x cannot be equal to (-1).

3) if p odd and q even then $p = 2n+1$ and $q = 2m$

$$x^3 + x = \frac{(2n+1)^3}{(2m)^3} + \frac{2n+1}{2m} = \frac{(2n+1)^3 + (2m)^2(2n+1)}{(2m)^3}$$

So numerator must be odd and denominator is even.

So $(x^3 + x)$ cannot be equal to (-1).

So in all 3 cases $\therefore$ r cannot be a rational number.