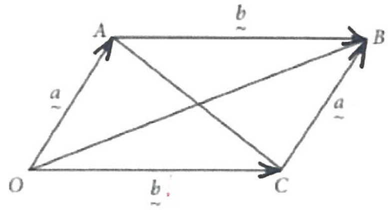
VECTORS IN GEOMETRIC PROOFS

1 Consider the parallelogram $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{b}$. Express $\overrightarrow{OB}$ and $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Hence show that the diagonals of a parallelogram meet at right angles if and only if it is a rhombus.



$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = \vec{a} - \vec{b}$

If OABC is a rhombus, then $|\vec{a}| = |\vec{b}|$

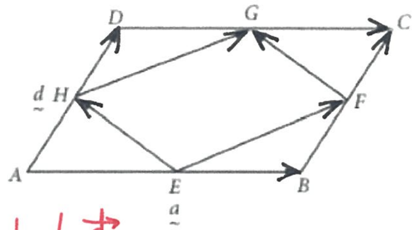
In that case $\overrightarrow{OB} \cdot \overrightarrow{CA} = (\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = \vec{a}.(-\vec{b}) + \vec{a}.\vec{a} + \vec{b}.\vec{a} + \vec{b}.(-\vec{b})$

So $\overrightarrow{OB}.\overrightarrow{CA} = |\vec{a}||\vec{a}| \cos(\pi - \theta) + |\vec{a}||\vec{a}| + |\vec{a}||\vec{a}| \cos\theta + |\vec{a}||\vec{a}| \cos\pi$

$\qquad = |\vec{a}|^2 [-\cos\theta + 1 + \cos\theta - 1]$ where $\theta = \angle(\vec{a}, \vec{b})$

$\qquad = 0$ So $\overrightarrow{OB}$ and $\overrightarrow{CA}$ are then perpendicular.

2 Consider the parallelogram $ABCD$, where $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AD} = \underset{\sim}{d}$. Prove that the midpoints of the sides of a parallelogram join to form a parallelogram.



~~$\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\vec{a}$~~

$\overrightarrow{EH} = \overrightarrow{EA} + \overrightarrow{AH} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AD} = -\frac{1}{2}\vec{a} + \frac{1}{2}\vec{d}$

$\overrightarrow{FG} = \overrightarrow{FC} + \overrightarrow{CG} = \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CD} = \frac{1}{2}\vec{d} - \frac{1}{2}\vec{a}$

So $\overrightarrow{EH} = \overrightarrow{FG}$

Likewise:

$\overrightarrow{HG} = \overrightarrow{HD} + \overrightarrow{DG} = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}\vec{d} + \frac{1}{2}\vec{a} = \frac{1}{2}(\vec{a}+\vec{d})$

$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{d} = \frac{1}{2}(\vec{a}+\vec{d})$

So $\overrightarrow{HG} = \overrightarrow{EF}$

$\therefore$ The two opposites sides of the quadrilateral are of equal length and parallel to each other.

$\therefore$ EHGF is a //gram.

3 Consider the quadrilateral $ABCD$, as shown.

Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$, $\overrightarrow{CD} = \underset{\sim}{c}$ and $\overrightarrow{DA} = \underset{\sim}{d}$.

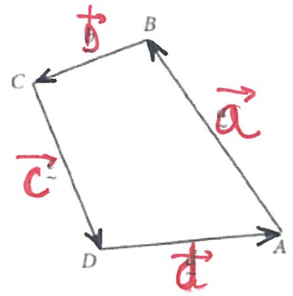
Which one of the following statements is correct?

A $\underset{\sim}{a} - \underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{d}$ **B** $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

C $\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{d}$ $\boxed{\textbf{D}}$ $\underset{\sim}{a} + \underset{\sim}{c} = -\underset{\sim}{b} - \underset{\sim}{d}$

$\vec{a} + \vec{b} = -(\vec{c} + \vec{d})$

So $\vec{a} + \vec{c} = -\vec{b} - \vec{d}$

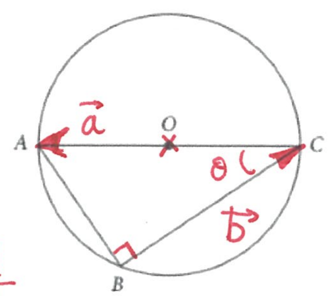


4 Consider the circle with centre O and radius $\overrightarrow{OA} = \underset{\sim}{a}$. B and C are points on the circle and $\overrightarrow{BC} = \underset{\sim}{b}$.

Which one of the following statements must be true?

A $\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$ NO **B** $\underset{\sim}{a} = -\frac{1}{2}\underset{\sim}{b}$ NO (not same direction)

C $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{b}$ **D** $2\underset{\sim}{a} \bullet \underset{\sim}{b} = -\underset{\sim}{b} \bullet \underset{\sim}{b}$

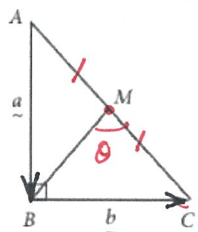


$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\pi - \theta)$ But $\cos\theta = \frac{|\vec{b}|}{2|\vec{a}|} = \frac{adj}{hyp}$

So $\vec{a} \cdot \vec{b} = -|\vec{a}| \frac{|\vec{b}||\vec{b}|}{2|\vec{a}|} = -\frac{|\vec{b}|^2}{2}$

So $2\vec{a} \cdot \vec{b} = -|\vec{b}|^2 = -\vec{b} \cdot \vec{b}$ $\boxed{D}$

5 Use vector methods to prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from all vertices.



$|\overrightarrow{MA}| = |\overrightarrow{MC}|$ as M is midpoint of $\overrightarrow{AC}$

We want to show that $|\overrightarrow{MB}| = |\overrightarrow{MA}| = |\overrightarrow{MC}|$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$ as ΔABC is right-angled at B.

So $(\overrightarrow{AM} + \overrightarrow{MB}) \cdot (\overrightarrow{BM} + \overrightarrow{MC}) = 0$

or $\overrightarrow{AM} \cdot \overrightarrow{BM} + \overrightarrow{AM} \cdot \overrightarrow{MC} + \overrightarrow{MB} \cdot \overrightarrow{BM} + \overrightarrow{MB} \cdot \overrightarrow{MC} = 0$

But $\overrightarrow{AM} = \overrightarrow{MC}$ and $\overrightarrow{MB} = -\overrightarrow{BM}$, ~~and~~

So $|\overrightarrow{AM}||\overrightarrow{BM}| \cos(\pi - \theta) + |\overrightarrow{AM}|^2 - |\overrightarrow{BM}|^2 + |\overrightarrow{MB}||\overrightarrow{MC}| \cos\theta = 0$

$-|\overrightarrow{AM}||\overrightarrow{BM}| \cos\theta + |\overrightarrow{AM}|^2 - |\overrightarrow{BM}|^2 + |\overrightarrow{MB}||\overrightarrow{AM}| \cos\theta = 0$

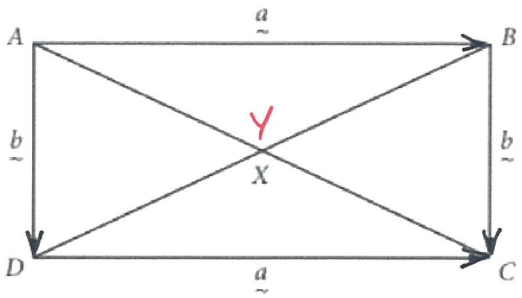
So $|\overrightarrow{AM}|^2 = |\overrightarrow{BM}|^2$

So $|\overrightarrow{AM}| = |\overrightarrow{BM}|$

7 *ABCD* is a rectangle.

(a) Prove that the diagonals of a rectangle bisect each other.

(b) Prove that the diagonals of a rectangle are equal in length.



a) let X be the midpoint of BD
and Y be the midpoint of AC
we need to demonstrate that $X = Y$

$\vec{AY} = \frac{1}{2}\vec{AC} = \frac{1}{2}\left(\vec{AD} + \vec{DB} + \vec{BC}\right) = \frac{1}{2}\left(\vec{b} + \vec{DB} + \vec{b}\right) = \vec{b} + \frac{1}{2}\vec{DB}$

whereas $\vec{AX} = \vec{AD} + \vec{DX} = \vec{b} + \frac{1}{2}\vec{DB}$

$\therefore \vec{AX} = \vec{AY}$ $\therefore X = Y$ as both vectors have the same tail.

b) $|\vec{AC}| = |\vec{AB} + \vec{BC}| = |\vec{a} + \vec{b}|$

whereas $|\vec{DB}| = |\vec{DC} + \vec{CB}| = |\vec{a} + (-\vec{b})|$

$|\vec{AC}|^2 = [|\vec{a} + \vec{b}|]^2 = (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{b}.\vec{a} + |\vec{b}|^2$

But $\vec{a}$ and $\vec{b}$ are perpendicular, $\therefore \vec{a}.\vec{b} = \vec{b}.\vec{a} = 0$

So $|\vec{AC}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

$|\vec{DB}|^2 = [|\vec{a} + (-\vec{b})|]^2 = (\vec{a} - \vec{b}).(\vec{a} - \vec{b}) = |\vec{a}|^2 - \vec{a}.\vec{b} - \vec{b}.\vec{a} + |\vec{b}|^2$

But $\vec{a}$ and $\vec{b}$ are perpendicular, $\therefore \vec{a}.\vec{b} = \vec{b}.\vec{a} = 0$

So $|\vec{DB}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

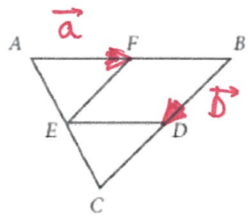
$\therefore |\vec{AC}|^2 = |\vec{DB}|^2$ $\therefore |\vec{AC}| = |\vec{DB}|$

8 $BDEF$ is a parallelogram contained within a triangle ABC, as shown.

Let $\overrightarrow{AF} = \underset{\sim}{a}$, $\overrightarrow{BD} = \underset{\sim}{b}$ and F be the midpoint of $\overline{AB}$.

(a) Find the vector $\overrightarrow{AE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(b) Use vector methods to prove that $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$.



$\overrightarrow{AF} = \vec{a}$ $\quad \overrightarrow{BD} = \vec{b}$ $\quad \overrightarrow{AF} = \frac{1}{2}\overrightarrow{AB}$

a) $\overrightarrow{AE} = \overrightarrow{AF} + \overrightarrow{FE}$

$\overrightarrow{AE} = \vec{a} + \overrightarrow{BD}$ (as BDEF is a parallelogram $\therefore \overrightarrow{FE} = \overrightarrow{BD}$)

So $\overrightarrow{AE} = \vec{a} + \vec{b}$

b) We know that ① $\overrightarrow{AF} = \frac{1}{2}\overrightarrow{AB}$ and ② BDEF is a //gram

Method ①: with similar triangles: $EF \parallel BD$ so $EF \parallel BC$ $\therefore \triangle AEF \,|||\, \triangle ABC$

So $\frac{AF}{EF} = \frac{AB}{BC}$ but $AF = \frac{1}{2}AB$ $\therefore \frac{1}{2EF} = \frac{1}{BC}$ so $2EF = BC$

But $BD = EF$ as BDEF is a //gram so $2BD = BC$

As BD and BC have a point in common, and as D is on the line BC we can say that $2\overrightarrow{BD} = \overrightarrow{BC}$ so $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$.

Method ②: with vectors. A, E and C are colinear $\therefore \overrightarrow{AC} = x\,\overrightarrow{AE}$

Likewise B, D and C are colinear $\therefore \overrightarrow{BC} = y\,\overrightarrow{BD}$

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = x\,\overrightarrow{AE} - y\,\overrightarrow{BD} = x\left[\vec{a} + \vec{b}\right] - y\,\vec{b}$

So $\overrightarrow{AB} = x\,\vec{a} + (x - y)\,\vec{b}$

But we also know that $\overrightarrow{AB} = 2\vec{a}$.

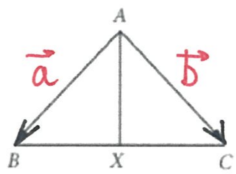
$\therefore$, equalling both quantities on each sides $\begin{cases} x = 2 \\ x - y = 0 \end{cases}$ $\begin{cases} x = 2 \\ y = 2 \end{cases}$

So $\overrightarrow{BC} = 2\,\overrightarrow{BD}$ or $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$

9 ABC is a triangle with $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{b}$. X is the midpoint of BC as shown.

(a) Find $\overrightarrow{BX}$ and $\overrightarrow{AX}$.

(b) Find $2\left(\overrightarrow{BX} \bullet \overrightarrow{BX} + \overrightarrow{AX} \bullet \overrightarrow{AX}\right)$.



Apollonius' theorem relates to the length of a median of a triangle to the lengths of its sides. In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

(c) Prove this, i.e. prove that $\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 2\left(\left|\overrightarrow{AX}\right|^2 + \left|\overrightarrow{BX}\right|^2\right)$.

a) $\overrightarrow{BX} = \frac{1}{2}\overrightarrow{BC}$ as X is the midpoint of BC

But $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\vec{a} + \vec{b} = \vec{b} - \vec{a}$ so $\overrightarrow{BX} = \frac{1}{2}\left(\vec{b} - \vec{a}\right)$

$\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{BX} = \vec{a} + \frac{1}{2}\left(\vec{b} - \vec{a}\right) = \frac{\vec{a} + \vec{b}}{2}$

b) $2\left(\overrightarrow{BX}.\overrightarrow{BX} + \overrightarrow{AX}.\overrightarrow{AX}\right) = 2\left[|\overrightarrow{BX}|^2 + |\overrightarrow{AX}|^2\right]$

But also $\overrightarrow{BX}.\overrightarrow{BX} = \left[\frac{1}{2}(\vec{b} - \vec{a})\right] \cdot \left[\frac{1}{2}(\vec{b} - \vec{a})\right] = \frac{1}{4}\left[(\vec{b} - \vec{a})(\vec{b} - \vec{a})\right]$

So $\overrightarrow{BX}.\overrightarrow{BX} = \frac{1}{4}\left[|\vec{b}|^2 - \vec{b}.\vec{a} - \vec{a}.\vec{b} + |\vec{a}|^2\right] = \frac{1}{4}\left[|\vec{b}|^2 - 2\vec{a}.\vec{b} + |\vec{a}|^2\right]$

whereas $\overrightarrow{AX}.\overrightarrow{AX} = \left[\frac{1}{2}(\vec{a} + \vec{b})\right].\left[\frac{1}{2}(\vec{a} + \vec{b})\right] = \frac{1}{4}\left[(\vec{a} + \vec{b})(\vec{a} + \vec{b})\right]$

So $\overrightarrow{AX}.\overrightarrow{AX} = \frac{1}{4}\left[|\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2\right]$

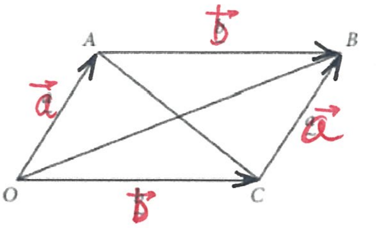
$\therefore 2\left(\overrightarrow{BX}.\overrightarrow{BX} + \overrightarrow{AX}.\overrightarrow{AX}\right) = 2 \times \left[\frac{1}{4}\left(|\vec{b}|^2 - 2\vec{a}.\vec{b} + |\vec{a}|^2\right) + \frac{1}{4}\left(|\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2\right)\right]$

$= |\vec{a}|^2 + |\vec{b}|^2$

c) $\therefore \quad |\vec{a}|^2 + |\vec{b}|^2 = 2\left[|\overrightarrow{BX}|^2 + |\overrightarrow{AX}|^2\right]$

or $\quad |\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2\left[|\overrightarrow{BX}|^2 + |\overrightarrow{AX}|^2\right]$

10 Consider the parallelogram $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{b}$.



- **(a)** Find the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- **(b)** Find the sum of the squares of the lengths of the sides of the parallelogram in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- **(c)** Find the sum of the squares of the lengths of the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- **(d)** Hence prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

a) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$ $\quad\quad$ $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\vec{a} + \vec{b}$

b) That is : $2\left[|\vec{a}|^2 + |\vec{b}|^2\right]$

c) $|\overrightarrow{OB}|^2 + |\overrightarrow{AC}|^2 = \overrightarrow{OB}\cdot\overrightarrow{OB} + \overrightarrow{AC}\cdot\overrightarrow{AC}$

$= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) + (\vec{b}-\vec{a})(\vec{b}-\vec{a})$

$= |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + |\vec{b}|^2 + |\vec{b}|^2 - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + |\vec{a}|^2$

$= 2|\vec{a}|^2 + 2|\vec{b}|^2 = 2\left[|\vec{a}|^2 + |\vec{b}|^2\right]$

d) $\therefore$ from b) and c)

$|\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CO}|^2 = |\overrightarrow{OB}|^2 + |\overrightarrow{AC}|^2$