PARTIAL FRACTIONS, LINEAR FACTORS

2 Reduce each rational function to its partial fractions.

(a) $\dfrac{4}{(x-1)(x+3)}$ **(b)** $\dfrac{2x-1}{(x+2)(x-3)}$

a) $$\frac{4}{(x-1)(x+3)} = \frac{a}{(x-1)} + \frac{b}{(x+3)} = \frac{x(a+b)+(3a-b)}{(x-1)(x+3)}$$

so $\begin{cases} a+b=0 \\ 3a-b=4 \end{cases}$ $\begin{cases} b=-a \\ 3a+a=4 \end{cases}$ $\begin{cases} a=1 \\ b=-1 \end{cases}$

$$\therefore \frac{4}{(x-1)(x+3)} = \frac{1}{x-1} - \frac{1}{x+3}$$

b) $$\frac{2x-1}{(x+2)(x-3)} = \frac{a}{x+2} + \frac{b}{x-3}$$

$$= \frac{x[a+b]+[2b-3a]}{(x+2)(x-3)}$$

So $\begin{cases} a+b=2 \\ 2b-3a=-1 \end{cases} \iff \begin{cases} 2a+2b=4 \\ -3a+2b=-1 \end{cases} \iff \begin{cases} -3a-2a=-1-4 \\ a+b=2 \end{cases}$

So $-5a=-5$ i.e. $a=1$

and $b=2-a=2-1=1$

$$\therefore \frac{2x-1}{(x+2)(x-3)} = \frac{1}{x+2} + \frac{1}{x-3}$$

(c) $\dfrac{3x+1}{x(x+1)}$ (d) $\dfrac{2x^2-6x-2}{x(x-1)(x+2)}$

c) $\dfrac{3x+1}{x(x+1)} = \dfrac{a}{x} + \dfrac{b}{x+1} = \dfrac{a(x+1)+bx}{x(x+1)}$

$\therefore \begin{cases} a+b=3 \\ a=1 \end{cases} \Longleftrightarrow \begin{cases} a=1 \\ b=2 \end{cases}$

$\therefore \dfrac{3x+1}{x(x+1)} = \dfrac{1}{x} + \dfrac{2}{x+1}$

d) $\dfrac{2x^2-6x-2}{x(x-1)(x+2)} = \dfrac{a}{x} + \dfrac{b}{x-1} + \dfrac{c}{x+2}$

$\_\_\_ = \dfrac{a(x-1)(x+2)+b(x+2)x+c\,x(x-1)}{x(x-1)(x+2)}$

$\_\_\_ = \dfrac{a(x^2+x-2)+b(x^2+2x)+c(x^2-x)}{x(x-1)(x+2)}$

$\_\_\_ = \dfrac{x^2[a+b+c]+x[a+2b-c]+[-2a]}{x(x-1)(x+2)}$

$\therefore -2=-2a$ so $a=1$

$\begin{cases} 1+b+c=2 \\ 1+2b-c=-6 \end{cases} \Longleftrightarrow \begin{cases} b+c=1 \\ 2b-c=-7 \end{cases} \Longleftrightarrow \begin{cases} 3b=-6 \\ b+c=1 \end{cases}$

$\Longleftrightarrow \begin{cases} b=-2 \\ c=1-b=1+2=3 \end{cases}$

$\dfrac{2x^2-6x-2}{x(x-1)(x+2)} = \dfrac{1}{x} - \dfrac{2}{x-1} + \dfrac{3}{x+2}$

PARTIAL FRACTIONS, LINEAR FACTORS

(a) $\dfrac{3x-19}{(x+3)(2x-1)}$ (b) $\dfrac{5x}{x^2+x-6}$

a) $\dfrac{3x-19}{(x+3)(2x-1)} = \dfrac{a}{x+3} + \dfrac{b}{2x-1} = \dfrac{x(2a+b)+(-a+3b)}{(x+3)(2x-1)}$

So $\begin{cases} 2a+b=3 \\ -a+3b=-19 \end{cases} \Leftrightarrow \begin{cases} 2a+b=3 \\ -2a+6b=-38 \end{cases} \Leftrightarrow \begin{cases} 7b=-35 \\ 2a+b=3 \end{cases}$

$\Leftrightarrow \begin{cases} b=-5 \\ 2a-5=3 \end{cases} \Leftrightarrow \begin{cases} b=-5 \\ 2a=8 \end{cases}$ so $a=4$

$$\frac{3x-19}{(x+3)(2x-1)} = \frac{4}{x+3} - \frac{5}{2x-1}$$

b) For the quadratic: $\Delta = 1-4\times(-6) = 25$

so 2 roots $x = \dfrac{-1\pm 5}{2}$ so (-3) and 2

$\dfrac{5x}{x^2+x-6} = \dfrac{a}{x+3} + \dfrac{b}{x-2} = \dfrac{x(a+b)+(3b-2a)}{(x+3)(x-2)}$

$\begin{cases} a+b=5 \\ 3b-2a=0 \end{cases} \Leftrightarrow \begin{cases} 2a+2b=10 \\ -2a+3b=0 \end{cases} \Leftrightarrow \begin{cases} 5b=10 \\ a+b=5 \end{cases}$

$\Leftrightarrow b=2$ and $a=5-b=5-2=3$

So $\dfrac{5x}{x^2+x-6} = \dfrac{3}{x+3} + \dfrac{2}{x-2}$

(e) $\frac{3(3x+1)}{x^2-9}$ **(f)** $\frac{1-2x}{2x^2+7x+6}$

e) $$\frac{3(3x+1)}{(x-3)(x+3)} = \frac{a}{x-3} + \frac{b}{x+3} = \frac{x(a+b)+(3a-3b)}{(x-3)(x+3)}$$

So $\begin{cases} a+b=9 \\ 3a-3b=3 \end{cases} \Leftrightarrow \begin{cases} a+b=9 \\ a-b=1 \end{cases} \Leftrightarrow \begin{cases} 2a=10 \\ a+b=9 \end{cases}$

$\Leftrightarrow \begin{cases} a=5 \\ b=9-a=9-5=4 \end{cases}$

$$\frac{3(3x+1)}{(x-3)(x+3)} = \frac{5}{x-3} + \frac{4}{x+3}$$

f) $\Delta = 49 - 4 \times 6 \times 2 = 1$ $\quad x = \frac{-7 \pm 1}{4}$ $\quad x=-2$ or $x=-\frac{3}{2}$

So $2x^2+7x+6 = 2(x+2)\left(x+\frac{3}{2}\right) = (x+2)(2x+3)$

$$\frac{1-2x}{2x^2+7x+6} = \frac{a}{x+2} + \frac{b}{2x+3} = \frac{x(2a+b)+(2b+3a)}{2x^2+7x+6}$$

So $\begin{cases} 2a+b=-2 \\ 2b+3a=1 \end{cases} \Leftrightarrow \begin{cases} 4a+2b=-4 \\ 3a+2b=1 \end{cases} \Leftrightarrow \begin{cases} a=-5 \\ b=-2-2a=-2+10 \\ \quad =8 \end{cases}$

$$\frac{1-2x}{2x^2+7x+6} = \frac{-5}{x+2} + \frac{8}{2x+3}$$

PARTIAL FRACTIONS, LINEAR FACTORS

(g) $\dfrac{2x^2+x+6}{x^2-4}$ (h) $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

g) the degree of the numerator is the same than the denominator so we can do some simplification first.

$$\frac{2x^2+x+6}{x^2-4} = \frac{2(x^2-4)+8+x+6}{x^2-4} = 2 + \frac{x+14}{x^2-4}$$

$$\frac{x+14}{(x-2)(x+2)} = \frac{a}{x-2} + \frac{b}{x+2} = \frac{x(a+b)+2a-2b}{x^2-4}$$

$$\begin{cases} a+b=1 \\ 2a-2b=14 \end{cases} \iff \begin{cases} 2a+2b=2 \\ 2a-2b=14 \end{cases} \iff \begin{cases} 4a=16 \\ a+b=1 \end{cases} \iff \begin{cases} a=4 \\ b=-3 \end{cases}$$

So $\dfrac{2x^2+x+6}{x^2-4} = 2 + \dfrac{4}{x-2} - \dfrac{3}{x+2}$

h) $x(x-1)(2x+3) = 2x^3+x^2-3x$

So $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)} = \dfrac{(2x^3+x^2-3x)+3x-x-3}{2x^3+x^2-3x}$

$\_\_\_ = 1 + \dfrac{2x-3}{x(x-1)(2x+3)}$

$\_\_ = 1 + \dfrac{a}{x} + \dfrac{b}{x-1} + \dfrac{c}{2x+3} = 1 + \dfrac{a(x-1)(2x+3)+bx(2x+3)+cx(x-}{x(x-1)(2x+3)}$

$$a(x-1)(2x+3)+bx(2x+3)+cx(x-1) = x^2(2a+2b+c)+x(a+3b-c)-3a$$

So $-3a=-3$ i.e. $a=1$

and $\begin{cases} 2+2b+c=0 \\ 1+3b-c=2 \end{cases} \iff \begin{cases} 2b+c=-2 \\ 3b-c=1 \end{cases} \iff \begin{cases} 5b=-1 \\ c=3b-1 \end{cases}$ $b=-1/5$

$c = -\dfrac{3}{5} - 1 = -\dfrac{8}{5}$

So $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)} = 1 + \dfrac{1}{x} - \dfrac{1}{5(x-1)} - \dfrac{8}{5(2x+3)}$

PARTIAL FRACTIONS, LINEAR FACTORS

7 Reduce $\dfrac{5x^2+26x+29}{x^3+6x^2+11x+6}$ to its partial fractions.

$x=-2$ is an obvious root of the denominator.

$$
\begin{array}{r|l}
 & x^2+4x+3 \\
\hline
x+2 & x^3+6x^2+11x+6 \\
 & x^3+2x^2 \\
\hline
 & \quad 4x^2+11x+6 \\
 & \quad 4x^2+8x \\
\hline
 & \qquad 3x+6
\end{array}
$$

So $x^3+6x^2+11x+6=(x+2)(x^2+4x+3)$

then (-1) is an obvious root of the quadratic

$\text{\_\_\_\_\_\_} = (x+2)(x+1)(x+3)$

Then: $\dfrac{5x^2+26x+29}{(x+2)(x+1)(x+3)} = \dfrac{a}{x+2}+\dfrac{b}{x+1}+\dfrac{c}{x+3}$

$\text{\_\_\_\_\_\_} = \dfrac{a(x+1)(x+3)+b(x+2)(x+3)+c(x+2)(x+1)}{\text{denominator}}$

$\text{\_\_\_\_} = \dfrac{x^2(a+b+c)+x(4a+5b+3c)+(3a+6b+2c)}{\text{denominator}}$

So $\begin{cases} a+b+c=5 & ① \\ 4a+5b+3c=26 & ② \\ 3a+6b+2c=29 & ③ \end{cases} \iff \begin{cases} 2a+2b+2c=10 & ① \\ 3a+6b+2c=29 & ③ \\ 4a+5b+3c=26 & ② \end{cases}$

① - ③ $\Rightarrow -a-4b=-19$ so $a=19-4b$

So ② becomes $4(19-4b)+5b+3c=26$ or $-11b+3c=-50$

① is $(19-4b)+b+c=5$ or $-3b+c=-14$ or $-9b+3c=-42$

So $-2b=-8$ $\boxed{b=4}$ then $a=19-4\times4=3$

and $c=5-a-b=5-3-4=-2$

$$\frac{5x^2+26x+29}{x^3+6x^2+11x+6}=\frac{3}{x+2}+\frac{4}{x+1}-\frac{2}{x+3}$$