UNIFORM PROBABILITY DISTRIBUTION

1 A number from 1 to 16 is chosen at random. The random variable, R, represents the value chosen. Find the following values:

(a) $E(R)$ **(b)** $\mathrm{Var}(R)$

$$E(R) = \frac{n+1}{2}$$

$$E(R) = \frac{16+1}{2}$$

$$E(R) = 8.5$$

$$\mathrm{Var}(R) = \frac{n^2-1}{12} = \frac{16^2-1}{12}$$

$$\mathrm{Var}(R) = 21.25$$

2 A cleaner has nine similar-looking keys on a key chain. He tries them in turn, until he finds the one that opens the lock.

(a) What is the expected number of attempts for the cleaner to open the lock?

(b) What is the variance of the number of attempts?

a) $E(x) = \frac{n+1}{2} = \frac{9+1}{2} = \frac{10}{2} = 5$ attempts on average to find the right key

b) $\mathrm{Var}(x) = \frac{n^2-1}{12} = \frac{9^2-1}{12} = \frac{80}{12} = 6\frac{2}{3}$

3 A die in the shape of a tetrahedron (a solid with four triangular faces) is rolled. The four faces are numbered 1 to 4. Let F be a random variable that represents the value that is face down on the table.

(a) What is the expected value, $E(F)$? **(b)** What is the variance, $\mathrm{Var}(F)$?

$$E(F) = \frac{n+1}{2}$$

$$E(F) = \frac{4+1}{2}$$

$$E(F) = \frac{5}{2} = 2.5$$

$$\mathrm{Var}(F) = \frac{n^2-1}{12}$$

$$\mathrm{Var}(F) = \frac{4^2-1}{12}$$

$$\mathrm{Var}(F) = \frac{15}{12} = 1.25$$

4 A spinner is equally divided into n segments and each segment contains a value from 1 to n.

(a) If the expected value is 11.5, then n is equal to which of the following values?

A 20 **B** 21 $\boxed{\textbf{C}\ 22}$ **D** 23

$E(X) = 11.5 = \frac{n+1}{2}$ so $n+1 = 23$

$n = 22$

(b) If the variance is 14 then n is equal to which of the following values?

A 9 **B** 10 **C** 12 $\boxed{\textbf{D}\ 13}$

$Var(X) = \frac{n^2-1}{12} = 14$

so $n^2 - 1 = 168$

$n^2 = 169$

$n = 13$

5 Consider the random variable X that has the following probability distribution:

x	1	2	3	4	5
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

(a) Find the following values:

(i) $E(X)$ **(ii)** $\mathrm{Var}(X)$

(b) Now consider the random variable Y that has the following probability distribution:

y	2	3	4	5	6
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

Find, from first principles, the following values:

(i) $E(Y)$ **(ii)** $\mathrm{Var}(Y)$

(c) What can you say about the values of $E(X)$ and $E(Y)$?

(d) What can you say about the values of $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$?

(e) Now consider the random variable Z that has the following probability distribution:
Without doing any additional calculations, determine the following values:

z	6	7	8	9	10
$P(Z = Z)$	0.2	0.2	0.2	0.2	0.2

(i) $E(Z)$ **(ii)** $\mathrm{Var}(Z)$

a) i) $E(X) = \frac{n+1}{2} = \frac{5+1}{2} = 3$ ii) $Var(X) = \frac{n^2-1}{12} = \frac{5^2-1}{12} = \frac{24}{12} = 2$

b) i) $E(Y) = 2\times0.2 + 3\times0.2 + \dots + 6\times0.2 = 0.2\times(2+3+4+5+6) = 4$

ii) $Var(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = 2^2\times0.2 + 3^2\times0.2 + \dots + 6^2\times0.2 = 0.2\times(2^2+3^2+4^2+5^2+6^2)$

$E(Y^2) = 18$ so $Var(Y) = 18 - 4^2 = 2$

c) $Y = X + 1$ and $E(Y) = E(X) + 1$

d) $Var(Y) = Var(X)$

e) i) ~~E(Z)~~ $Z = X + 5$

So $E(Z) = E(X) + 5 = 3 + 5 = 8$

and $Var(Z) = Var(X) = 2$

UNIFORM PROBABILITY DISTRIBUTION

7 A six-sided die numbered 1 to 6 is rolled twice and the values obtained are added together.

(a) Construct a probability distribution table for this event, using S to represent the variable.

(b) Draw a bar chart to illustrate the distribution.

(c) How would you describe this distribution?

(d) Calculate the following values:

(i) $E(S)$ **(ii)** $\mathrm{Var}(S)$

(e) Recall that for a single roll of such a die: $E(X) = 3\frac{1}{2}$ and $\mathrm{Var}(X) = 2\frac{11}{12}$. Comment on the relationship between the values for $E(X)$, $\mathrm{Var}(X)$, $E(S)$ and $\mathrm{Var}(S)$.

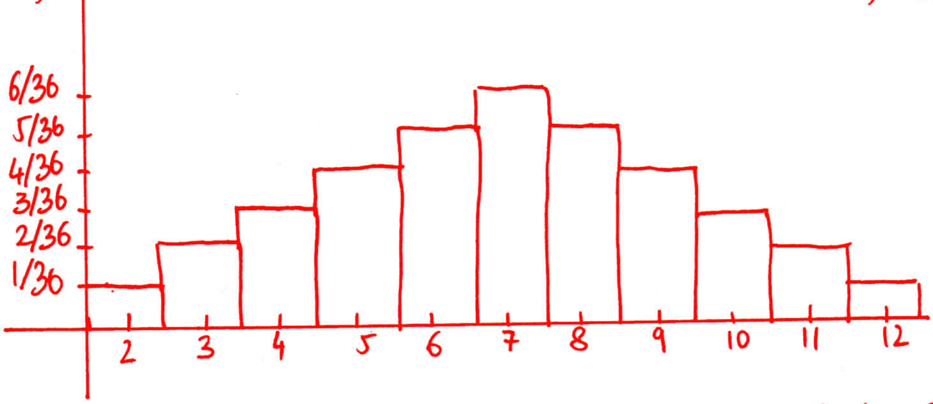
a)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

x_i	2	3	4	5	6	7	8	9	10
p	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$

x	11	12
$P(X=x_i)$	$\frac{2}{36}$	$\frac{1}{36}$

b)



c) symmetrical distribution

d) i) $E(S) = \frac{1}{36}\left(2\times1 + 3\times2 + 4\times3 + 5\times4 + 6\times5 + 7\times6 + 8\times5 + 9\times4 + 10\times3 + 11\times2 + 12\times1\right) = 7$

ii) $\mathrm{Var}(S) = E(S^2) - [E(S)]^2$

$E(S^2) = \frac{1}{36}\left[2^2\times1 + 3^2\times2 + 4^2\times3 + 5^2\times4 + \dots + 11^2\times2 + 12^2\times1\right] = \frac{329}{6}$

So $\mathrm{Var}(S) = \frac{329}{6} - (7)^2 = \frac{35}{6} = 5\,5/6$

e) $E(X) = 3.5$ so obviously $E(S) = 2\times3.5 = 2\times E(X)$

$\mathrm{Var}(X) = 2\frac{11}{12} = \frac{35}{12}$ so $\mathrm{Var}(S) = 2\,\mathrm{Var}(X)$

(f) Consider a spinner that has four equally-sized sections labelled 1 to 4. Let V be the value the spinner stops on. Calculate the following values:
(i) $E(V)$ **(ii)** $\mathrm{Var}(V)$

(g) The spinner is now spun twice. Let T be the sum of the two values obtained. Calculate the following values:
(i) $E(T)$ **(ii)** $\mathrm{Var}(T)$

(h) Comment on the relationship between the values for $E(V)$, $\mathrm{Var}(V)$, $E(T)$ and $\mathrm{Var}(T)$.

(i) Can you now make a general comment on the values obtained for the mean (expected value) and variance when two identical uniform variables, with values between 1 and n, are added?

f) i) This is a uniform distribution, so $E(V) = \frac{n+1}{2} = \frac{4+1}{2} = 2.5$

ii) $\mathrm{Var}(V) = \frac{n^2-1}{12} = \frac{4^2-1}{12} = \frac{15}{12} = 1.25$

g)

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

x_i	2	3	4	5	6	7	8
$P(T=x_i)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

i) $E(T) = 2\times\frac{1}{16} + 3\times\frac{2}{16} + 4\times\frac{3}{16} + 5\times\frac{4}{16} + 6\times\frac{3}{16} + 7\times\frac{2}{16} + 8\times\frac{1}{16} = 5$

ii) $\mathrm{Var}(T) = E(T^2) - [E(T)]^2$

$E(T^2) = \frac{1}{16}\left[2^2\times1 + 3^2\times2 + 4^2\times3 + 5^2\times4 + 6^2\times3 + 7^2\times2 + 8^2\times1\right] = 27.5$

So $\mathrm{Var}(T) = 27.5 - 5^2 = 2.5$

h) $E(T)$ is twice the value of $E(V)$, as T is the sum of two V. Same for $\mathrm{Var}(T)$. The expected value and the variance for the double spin are each double the respective single spin values.

i) When two identical uniform discrete variables are added, the expected value and variance are both double the single values.