QUADRATIC INEQUALITIES

Possible operations on inequalities:

- adding or subtracting the same number from both sides does not alter the inequality
- multiplying both sides of the inequality by a positive number does not alter the direction of the inequality
- multiplying both sides of the inequality by a negative number does change the direction of the inequality

  e.g.[1] $3 > 2$ but $-6 < -4$ after multiplying both sides of the inequality by (-2)

- taking the reciprocal of both sides of an inequality reverses its direction when both sides have the same sign, but not if the signs are different:

  e.g. $5 < 7$ but $\frac{1}{5} > \frac{1}{7}$

  $3 > -4$ and $\frac{1}{3} > -\frac{1}{4}$

Proof:

Let $a > b$, where a, b have the same sign.
Divide both sides by ab, which must be positive as a, b have the same sign:

$$\frac{a}{ab} > \frac{b}{ab}$$
$$\frac{1}{b} > \frac{1}{a}$$
$$\therefore \frac{1}{a} < \frac{1}{b}$$

Let $a > b$, where a, b have different signs.
Divide both sides by ab, which must be negative as a, b have different signs:

$$\frac{a}{ab} < \frac{b}{ab} \quad \text{(inequality changes)}$$
$$\frac{1}{b} < \frac{1}{a}$$
$$\therefore \frac{1}{a} > \frac{1}{b}$$

- Squaring both sides of an inequality produces a positive number of both sides, so the direction of the resulting inequality depends on which of the number had the largest absolute value.

  Thus if a and b are positive and $a > b$, then $a^2 > b^2$ (i.e.[2] you can square both sides of an inequality without changing direction if both sides are known to be positive).

Proof: Let $a > b$ and a, b both be positive.

Multiply the original inequality by a, which is positive: $a^2 > ab$

Multiply the original inequality by b, which is positive: $ab > b^2$

Link the two results together: $a^2 > ab > b^2$

$\therefore$ $a^2 > b^2$

- Square roots of both sides of an inequality is only defined if both sides of the inequality are non-negative. The direction of the inequality does not change (if a and b positive and $a > b$, then $\sqrt{a} > \sqrt{b}$)

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[1] e.g. is the abbreviation for the Latin phrase **exempli gratia**, meaning "for example,"
[2] i.e. is the abbreviation for the Latin phrase **id est**, meaning "that is,"

QUADRATIC INEQUALITIES

Quadratic inequalities

Example 1

Solve $x^2 - 4x > 0$.

Solution

To solve a quadratic inequality you must *not* simply factorise it like a quadratic equation, as this is wrong:

$x(x-4) > 0$

$x > 0,\ x - 4 > 0$

$x > 0,\ x > 4$

If you now substitute any negative number for x in the original inequality, you will see that it is a solution, so $x > 0$, $x > 4$ must not be the complete answer. Similarly, $x = 1$ is not a solution even though it is included in $x > 0$, $x > 4$, so this method must be wrong. What has happened? Where is the error?

The answer is that for $x(x-4) > 0$ to be true (i.e. for the product of two factors to be positive), both x and $x - 4$ must be positive *or* both factors must be negative.

The solution can proceed as follows:

$x(x-4) > 0$

$x > 0$ and $x - 4 > 0$ **or** $x < 0$ and $x - 4 < 0$

$x > 0$ and $x > 4$ | $x < 0$ and $x < 4$

For both $x > 0$ and $x > 4$ to be true: $x > 4$

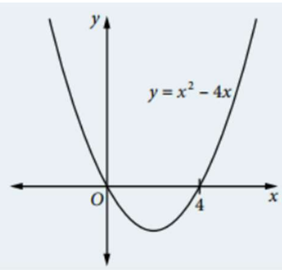
For both $x < 0$ and $x < 4$ to be true: $x < 0$

Hence the correct solution is $x < 0$, $x > 4$ (which is read as '$x < 0$ or $x > 4$').

Alternatively, a graphical method can be used:

- Sketch the parabola $y = x^2 - 4x$ (concave up, cuts the x-axis at 0 and 4).
- Identify the x values for which the parabola is above the x-axis (because you are looking for the places where $y > 0$). These values are the solution.

The graph shows that the solution is $x < 0$ or $x > 4$.



The graphical method in the example above has the advantage of providing a visual picture. It is also an easy method for higher-degree inequalities.

Polynomial inequalities

- use the factored form of the polynomial to sketch the graph of the function
- use the graph to identify the x values for which the graph is above (or below) the x-axis (depending of the inequality). These values are the solutions.
- if the inequality is $\leq$ or $\geq$, be careful to include the values where the graph cuts or touches the x-axis.

Example 2

Solve $x^2(x-2)(x+1) \geq 0$.

Solution

Sketch $y = x^2(x-2)(x+1)$, noting the double zero at $x = 0$ (a turning point on the x-axis), and zeros at $x = 2$ and $x = -1$.

Find where the graph is on or above the x-axis. The solution is $x \leq -1$, $x = 0$, $x \geq 2$.