TRIGONOMETRIC PRODUCTS AS SUMS OR DIFFERENCES

$$\boldsymbol{\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]}$$

Proof:

As demonstrated before: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

and $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Adding both equations, we obtain: $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

or by swapping both sides: $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

Therefore $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$$\boldsymbol{\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]}$$

Proof:

As demonstrated before: $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Subtracting both equations, we obtain: $\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$

or by swapping both sides: $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

Therefore $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$$\boldsymbol{\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]}$$

Proof:

As demonstrated before: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Therefore, adding both equations: $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

or by swapping both sides: $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

Therefore $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

TRIGONOMETRIC PRODUCTS AS SUMS OR DIFFERENCES

$$\boldsymbol{\sin\theta + \sin\varphi = 2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)}$$

Proof:

As demonstrated before: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Therefore: $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

Defining $\alpha+\beta=\theta$ and $\alpha-\beta=\varphi$

So $\alpha = \frac{\theta+\varphi}{2}$ and $\beta = \frac{\theta-\varphi}{2}$

Therefore $\sin\theta + \sin\varphi = 2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$

$$\boldsymbol{\sin\theta - \sin\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)}$$

Proof:

As demonstrated before: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Therefore: $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$

Defining $\alpha+\beta=\theta$ and $\alpha-\beta=\varphi$

So $\alpha = \frac{\theta+\varphi}{2}$ and $\beta = \frac{\theta-\varphi}{2}$

Therefore $\sin\theta - \sin\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$

TRIGONOMETRIC PRODUCTS AS SUMS OR DIFFERENCES

$$\boldsymbol{\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)}$$

Proof:

As demonstrated above: $cos(\alpha+\beta) = cos\,\alpha\,cos\,\beta - sin\,\alpha\,sin\,\beta$

$cos(\alpha-\beta) = cos\,\alpha\,cos\,\beta + sin\,\alpha\,sin\,\beta$

Therefore: $cos(\alpha+\beta) + cos(\alpha-\beta) = 2\,cos\,\alpha\,cos\,\beta$

Defining $\alpha+\beta=\theta$ and $\alpha-\beta=\varphi$

So $\alpha=\frac{\theta+\varphi}{2}$ and $\beta=\frac{\theta-\varphi}{2}$

Therefore: $cos\,\theta + cos\,\varphi = 2\,cos\left(\frac{\theta+\varphi}{2}\right)cos\left(\frac{\theta-\varphi}{2}\right)$

$$\boldsymbol{\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)}$$

Proof:

As demonstrated above: $cos(\alpha+\beta) = cos\,\alpha\,cos\,\beta - sin\,\alpha\,sin\,\beta$

$cos(\alpha-\beta) = cos\,\alpha\,cos\,\beta + sin\,\alpha\,sin\,\beta$

Therefore: $cos(\alpha+\beta) - cos(\alpha-\beta) = -2\,sin\,\alpha\,sin\,\beta$

Defining $\alpha+\beta=\theta$ and $\alpha-\beta=\varphi$

So $\alpha=\frac{\theta+\varphi}{2}$ and $\beta=\frac{\theta-\varphi}{2}$

Therefore: $cos\,\theta - cos\,\varphi = -2\,sin\left(\frac{\theta+\varphi}{2}\right)sin\left(\frac{\theta-\varphi}{2}\right)$

Example 11

Express each product as a sum or difference of trigonometric functions:

(a) $2\cos 5x \sin x$ **(b)** $2\sin 4A \sin A$ **(c)** $\cos 3\theta \cos 5\theta$ **(d)** $\sin 3\theta \cos\theta$

Solution

(a) $2\cos 5x \sin x = \sin(5x + x) - \sin(5x - x)$
$= \sin 6x - \sin 4x$

(b) $2\sin 4A \sin A = \cos(4A - A) - \cos(4A + A)$
$= \cos 3A - \cos 5A$

(c) $\cos 3\theta \cos 5\theta = \frac{1}{2}(\cos(3\theta + 5\theta) + \cos(3\theta - 5\theta))$
$= \frac{1}{2}(\cos 8\theta + \cos(-2\theta))$
$= \frac{1}{2}(\cos 8\theta + \cos 2\theta)$

(d) $\sin 3\theta \cos\theta = \frac{1}{2}(\sin(3\theta + \theta) + \sin(3\theta - \theta))$
$= \frac{1}{2}(\sin 4\theta + \sin 2\theta)$

Example 12

Convert the following sums or differences into products:

(a) $\sin 6x - \sin 4x$ **(b)** $\cos 3A - \cos 5A$ **(c)** $\cos 8\theta + \cos 2\theta$ **(d)** $\sin 3x + \sin x$

Solution

(a) $\sin 6x - \sin 4x = 2\cos\left(\frac{6x + 4x}{2}\right)\sin\left(\frac{6x - 4x}{2}\right)$
$= 2\cos 5x \sin x$

(b) $\cos 3A - \cos 5A = -2\sin\left(\frac{3A + 5A}{2}\right)\sin\left(\frac{3A - 5A}{2}\right)$
$= -2\sin 4A \sin(-A)$
$= 2\sin 4A \sin A$

(c) $\cos 8\theta + \cos 2\theta = 2\cos\left(\frac{8\theta + 2\theta}{2}\right)\cos\left(\frac{8\theta - 2\theta}{2}\right)$
$= 2\cos 5\theta \cos 3\theta$

(d) $\sin 3x + \sin x = 2\sin\left(\frac{3x + x}{2}\right)\cos\left(\frac{3x - x}{2}\right)$
$= 2\sin 2x \cos x$

Example 13

Show that $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$.

Solution

$$\text{LHS} = \frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta}$$
$$= \frac{(\sin 5\theta + \sin\theta) + \sin 3\theta}{(\cos 5\theta + \cos\theta) + \cos 3\theta}$$
$$= \frac{2\sin 3\theta \cos 2\theta + \sin 3\theta}{2\cos 3\theta \cos 2\theta + \cos 3\theta}$$
$$= \frac{\sin 3\theta(2\cos 2\theta + 1)}{\cos 3\theta(2\cos 2\theta + 1)}$$
$$= \frac{\sin 3\theta}{\cos 3\theta}$$
$$= \tan 3\theta = \text{RHS}$$