THE FACTOR THEOREM

When the remainder of a division is zero, the divisor is a factor of the expression being divided. For example, 2 is a factor of 14 because $14 = 2 \times 7 + 0$. Similarly, $(x + 3)$ is a factor of $x^2 + 4x + 3$ because $x^2 + 4x + 3 = (x + 3)(x + 1) + 0$.

The **factor theorem** states:

> For a polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

The converse of this result is also true:

> If $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$.

By finding the zeros of a polynomial (i.e. values of a such that $P(a) = 0$), you can factorise the polynomial. The first factor can usually be found by trial and error, and then by long division you can find the quotient, which will have a degree of one less than the original polynomial. You may then be able to find factors of the quotient which must also be factors of the original polynomial.

For example, if the original polynomial is cubic, then the quotient will be a quadratic polynomial, which you can then factorise into real factors (if they exist). For higher degree polynomials, further trial and error may be needed after long division until you can reach a factorisable quotient.

Example 8

$P(x) = x^3 - x^2 - 14x + 24$. Find one zero of the polynomial and then express $P(x)$ as a product of linear factors.

Solution

$P(x) = x^3 - x^2 - 14x + 24$ is monic, so the factors of 24 can be regarded as the only possible zeros, namely $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$:

$P(1) = 1 - 1 - 14 + 24 = 10 \neq 0$, $(x - 1)$ is not a factor

$P(-1) = -1 - 1 + 14 + 24 = 36 \neq 0$, $(x + 1)$ is not a factor

$P(2) = 8 - 4 - 28 + 24 = 0$, $(x - 2)$ is a factor

Hence $P(x) = (x - 2)(x^2 + x - 12)$

$= (x - 2)(x + 4)(x - 3)$

$$
\begin{array}{r}
x^2 + x - 12 \\
x - 2 \overline{)\, x^3 - x^2 - 14x + 24} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
x^2 - 14x \qquad \\
\underline{x^2 - 2x} \qquad \\
-12x + 24 \\
\underline{-12x + 24} \\
0
\end{array}
$$

The factor theorem could also have been used to find the zeros 3 and –4. You should use the method that you find easiest and quickest in each case.

Example 9

Find the linear factors of $x^4 + x^3 - 7x^2 - x + 6$.

Solution

$P(x) = x^4 + x^3 - 7x^2 - x + 6$

$P(1) = 1 + 1 - 7 - 1 + 6 = 0$ Hence $(x - 1)$ is a factor.

$P(-1) = 1 - 1 - 7 + 1 + 6 = 0$ Hence $(x + 1)$ is a factor.

You now know that $(x - 1)(x + 1) = x^2 - 1$ is a factor, so you can use this as the divisor in the long division.

Hence $P(x) = x^4 + x^3 - 7x^2 - x + 6 = (x - 1)(x + 1)(x^2 + x - 6)$

$= (x - 1)(x + 1)(x + 3)(x - 2)$

$$
\begin{array}{r}
x^2 + x - 6 \\
x^2 - 1 \overline{)\, x^4 + x^3 - 7x^2 - x + 6} \\
\underline{x^4 \qquad - x^2} \qquad\qquad \\
x^3 - 6x^2 - x \qquad \\
\underline{x^3 \qquad\quad - x} \qquad \\
-6x^2 \qquad + 6 \\
\underline{-6x^2 \qquad + 6} \\
0
\end{array}
$$