INTEGRALS INVOLVING TRIGONOMETRIC SUBSTITUTIONS

1 Evaluate $\int_{-4}^{4}\sqrt{16-x^2}\,dx$ using the substitution $x = 4\sin\theta$. What shape have you just found the area of?

$x = 4\sin\theta \quad \therefore \quad \frac{dx}{d\theta} = 4\cos\theta \quad$ or $\quad dx = 4\cos\theta\, d\theta$

When $x = -4 \quad \theta = -\pi/2$; when $x = 4$, $\theta = \pi/2$

$$\int_{-4}^{4}\sqrt{16-x^2}\,dx = \int_{-\pi/2}^{\pi/2}\sqrt{16-16\sin^2\theta} \times 4\cos\theta\, d\theta = \int_{-\pi/2}^{\pi/2} 4\cos\theta \times 4\cos\theta\, d\theta$$

$$= 16\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta = 16\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{16}{2}\left[\left[\theta\right]_{-\pi/2}^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}\right] = 8\left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right) = 8\pi$$

That's a semicircle of radius 4 , so area $= \frac{\pi r^2}{2} = \frac{\pi \times 16}{2} = 8\pi$ indeed.

2 Evaluate $\int_{0}^{1} x\sqrt{1-x^2}\,dx$ using the substitution $x = \sin\theta$.

$x = \sin\theta \quad$ so $\quad \frac{dx}{d\theta} = \cos\theta \quad\quad dx = \cos\theta\, d\theta$

when $x = 0 \quad \theta = 0$; when $x = 1$, $\theta = \pi/2$

$$\int_0^1 x\sqrt{1-x^2}\,dx = \int_0^{\pi/2}\sin\theta\sqrt{1-\sin^2\theta} \times \cos\theta\, d\theta = \int_0^{\pi/2}\sin\theta\cos^2\theta\, d\theta$$

$$= \left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = -\frac{\cos^3 \pi/2}{3} + \frac{\cos^3 0}{3} = \frac{1}{3}$$

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3 Evaluate $\int_0^2 \frac{dx}{4+x^2}$ using the substitution $x = 2\tan\theta$.

$x = 2\tan\theta$ so $\frac{dx}{d\theta} = 2\sec^2\theta$ $\quad dx = 2\sec^2\theta\, d\theta$

when $x = 0$, $\theta = 0$; when $x = 2$, $\theta = \pi/4$

$$\int_0^2 \frac{dx}{4+x^2} = \int_0^{\pi/4} \frac{2\sec^2\theta}{4(1+\tan^2\theta)}\, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan^2\theta}\, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^2\theta}$$

as $1+\tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$

$$\int_0^2 \frac{dx}{4+x^2} = \frac{1}{2}\int_0^{\pi/4} d\theta = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$$

6 Evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx$ using the substitution $x = \cos\theta$.

$x = \cos\theta$ so $\frac{dx}{d\theta} = -\sin\theta$ $\quad dx = -\sin\theta\, d\theta$

when $x = -\frac{1}{2}$ $\theta = \frac{2\pi}{3}$; when $x = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

$$\int_{-1/2}^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx = \int_{2\pi/3}^{\pi/3} \frac{\cos\theta}{\sin\theta} \times (-\sin\theta\, d\theta) = -\int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} \cos\theta\, d\theta$$

$$= \int_{\pi/3}^{2\pi/3} \cos\theta\, d\theta = \Big[\sin\theta\Big]_{\pi/3}^{2\pi/3}$$

$$= \sin\frac{2\pi}{3} - \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = 0$$

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9 Evaluate $\int_0^4 x\sqrt{16-x^2}\,dx$ using the substitution $x = 4\sin\theta$.

$x = 4\sin\theta \qquad \frac{dx}{d\theta} = 4\cos\theta \qquad$ so $dx = 4\cos\theta\, d\theta$

when $x = 0$, $\theta = 0$; when $x = 4$, $\theta = \pi/2$

$$\int_0^4 x\sqrt{16-x^2}\,dx = \int_0^{\pi/2} 4\sin\theta \times 4\cos\theta \times 4\cos\theta\, d\theta = 64\int_0^{\pi/2} \sin\theta\cos^2\theta\, d\theta$$

$$= 64\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{64}{3}\left[\cos^3\theta\right]_{\pi/2}^{0}$$

$$= \frac{64}{3}\left[1-0\right] = \frac{64}{3}$$

10 Evaluate $\int_0^{\sqrt{3}} \frac{1}{9+x^2}\,dx$ using the substitution $x = 3\tan\theta$.

$x = 3\tan\theta \qquad$ so $\frac{dx}{d\theta} = 3\sec^2\theta \qquad dx = 3\sec^2\theta\, d\theta$

when $x = 0$ $\theta = 0$; when $x = \sqrt{3}$, $\tan\theta = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}$

$$\int_0^{\sqrt{3}} \frac{dx}{9+x^2} = \int_0^{\pi/6} \frac{3\sec^2\theta\, d\theta}{9(1+\tan^2\theta)} = \frac{1}{3}\int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

as $1+\tan^2\theta = \sec^2\theta$

$$= \frac{1}{3}\int_0^{\pi/6} d\theta = \frac{1}{3}\times\left(\frac{\pi}{6}-0\right) = \frac{\pi}{18}$$

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12 Use the substitution $u = \tan x$ to find $\int \frac{e^{\tan x}}{\cos^2 x} dx$.

$u = \tan x \qquad \frac{du}{dx} = \sec^2 x \qquad du = \sec^2 x \, dx$

$\int \frac{e^{\tan x}}{\cos^2 x} dx = \int e^{\tan x} \times \sec^2 x \, dx = \int e^u \times du$

$\_\_\_\_ = e^u + C$

$\_\_\_\_ = e^{\tan x} + C$

$\therefore \int \frac{e^{\tan x}}{\cos^2 x} dx = e^{\tan x} + C$

13 Use the substitution $u = \cos x + \sin x$ to find $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$.

$u = \cos x + \sin x$

so $\frac{du}{dx} = -\sin x + \cos x$

$du = (-\sin x + \cos x)\,dx$

$\therefore \int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\int \frac{(-\sin x + \cos x)}{\sin x + \cos x}\,dx$

$= -\int \frac{du}{u} = -\ln|u| + C$

$\therefore \int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\ln|\cos x + \sin x| + C$

INTEGRALS INVOLVING TRIGONOMETRIC SUBSTITUTIONS

14 **(a)** By writing $\sec x = \dfrac{1}{\cos x}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

(b) Using the substitution $u = 1 + \sec x$, find $\int \dfrac{\sec x \tan x}{1 + \sec x} dx$.

a) $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{d}{dx}\left((\cos x)^{-1}\right) = (-1)(\cos x)^{-1-1} \times \dfrac{d}{dx}(\cos x)$

$\qquad = -\dfrac{1}{\cos^2 x} \times (-\sin x) = \dfrac{\sin x}{\cos^2 x} = \dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos x}$

$\therefore \dfrac{d}{dx}(\sec x) = \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x} = \sec x \times \tan x$

b) $u = 1 + \sec x$ so $\dfrac{du}{dx} = \dfrac{d}{dx}(\sec x) = \sec x \times \tan x$

$\therefore du = (\sec x \times \tan x)\, dx$

$\displaystyle\int \frac{\sec x \tan x}{1 + \sec x} dx = \int \frac{du}{u} = \ln|u| + C$

$\qquad = \ln|1 + \sec x| + C$

$\therefore \displaystyle\int \frac{\sec x \times \tan x}{1 + \sec x} dx = \ln|1 + \sec x| + C$