RATIONAL FUNCTION INEQUALITIES (x in denominator)

Example 3

Solve $\frac{1}{x-2} \geq -1$.

Solution

There are two important things to note here:

- $x \neq 2$, as the function $f(x) = \frac{1}{x-2}$ does not exist where $x = 2$
- if you multiply both sides by $(x - 2)$, you don't know whether $(x - 2)$ is positive or negative, so you don't know whether to reverse the inequality or not.

There are several possible methods to solve an inequality like this (see below). You should memorise the one or two methods that you find easiest and most useful, but you should be aware of the other methods too.

Method 1 (algebraic method requiring consideration of all possible cases)

$$\frac{1}{x-2} \geq -1$$

If $x > 2$, multiply by $(x - 2)$, which is positive:

$$1 \geq -(x-2)$$
$$1 \geq -x + 2$$
$$-1 \geq -x$$
$$1 \leq x \quad \text{(multiplying by } -1\text{)}$$
$$x \geq 1$$

Both $x > 2$ and $x \geq 1$ must be true.

This requires $x > 2$.

If $x < 2$, multiply by $(x - 2)$, which is negative:

$$1 \leq -(x-2) \quad \text{(note change of inequality)}$$
$$1 \leq -x + 2$$
$$-1 \leq -x$$
$$1 \geq x \quad \text{(multiplying by } -1\text{)}$$
$$x \leq 1$$

Both $x < 2$ and $x \leq 1$ must be true.

This requires $x \leq 1$.

Thus the complete solution is $x > 2$, $x \leq 1$.

Method 2 (algebraic method avoiding the need to consider different cases)

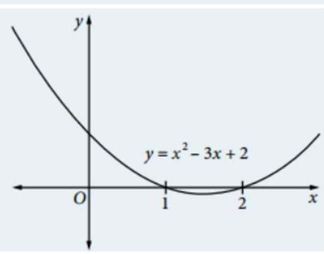
$$\frac{1}{x-2} \geq -1 \qquad \text{Note that } x \neq 2.$$

Multiply both sides by $(x - 2)^2$, which is known to be positive (so the inequality does not change):

$$x - 2 \geq -(x-2)^2$$
$$x - 2 \geq -x^2 + 4x - 4$$
$$x^2 - 3x + 2 \geq 0$$

You can now solve this quadratic inequality using the graphical method (as shown in Example 1).

The graph (at right) is on or above the x-axis for $x \leq 1$, $x \geq 2$. However, $x \neq 2$, so the solution is $x \leq 1$, $x > 2$.



Method 3 (variation on Method 2, producing an inequality in factorised form)

$$\frac{1}{x-2} \geq -1 \qquad \text{Note that } x \neq 2.$$

Make the right-hand side become zero, then simplify the resulting left-hand side:

$$\frac{1}{x-2} + 1 \geq 0$$
$$\frac{1 + (x-2)}{x-2} \geq 0$$
$$\frac{x-1}{x-2} \geq 0$$

Multiply both sides by $(x - 2)^2$:

$$(x-1)(x-2) \geq 0$$

This is the factorised form of the quadratic inequality obtained in Method 2. From this point on, the solution is identical to Method 2.

RATIONAL FUNCTION INEQUALITIES (x in denominator)

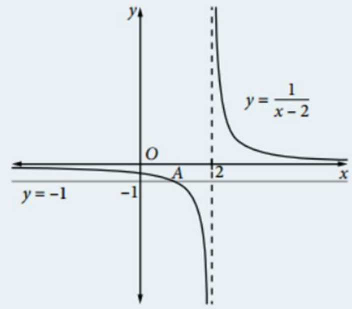
Method 4 (graphical method)

$$\frac{1}{x-2} \geq -1$$

Sketch a graph of $y = \frac{1}{x-2}$: it is a hyperbola with vertical asymptote at $x = 2$ and horizontal asymptote at the x-axis.

Also graph the horizontal line $y = -1$.

Use the graph to find the x values for which the hyperbola is on or above the horizontal line $y = -1$. You can see that this happens for all x values from point A back to the left, including point A, and for all x values to the right of the asymptote.



To find the x-coordinate of the point A, solve $\frac{1}{x-2} = -1$: the solution is $x = 1$. Thus the solution is $x \leq 1$, $x > 2$.

Method 5 (like the graphical method, but without drawing the graph)

$$\frac{1}{x-2} \geq -1$$

List all the x values that are excluded because they make the denominator become zero:

$x \neq 2$ (the vertical asymptote)

Solve the inequality's corresponding equation:

$$\frac{1}{x-2} = -1$$

$$1 = -x + 2$$

$x = 1$ This is the x-coordinate of the point of intersection of $y = \frac{1}{x-2}$ and $y = -1$.

The x values found by these two steps are 'critical values', to be marked on a number line. Use open circles to mark the excluded (asymptotic) values. If the inequality is < or >, use open circles to mark the x values from solving the equation; if the inequality is ≤ or ≥, use filled circles.

Now choose representative x values from each section of the number line and test to see whether the x values satisfy the original inequality:

Test $x = 0$: Is $\frac{1}{(0-2)} \geq -1$? Yes

Test $x = 1\frac{1}{2}$: Is $\frac{1}{1\frac{1}{2}-2} \geq -1$? No

Test $x = 3$: Is $\frac{1}{(3-2)} \geq -1$? Yes

This finds where the graph of $y = \frac{1}{x-2}$ is above the line $y = -1$. Thus the solution is $x \leq 1$, $x > 2$.

RATIONAL FUNCTION INEQUALITIES (x in denominator)

Example 4

Solve $\dfrac{1}{x^2 - x} > \dfrac{1}{x^2 - 1}$.

Solution

Using **Method 3** (from Example **3**):

$$\frac{1}{x(x-1)} > \frac{1}{(x-1)(x+1)}, \quad \text{so } x \neq 0, 1, -1$$

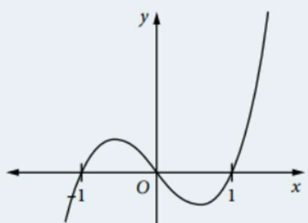
$$\frac{1}{x(x-1)} - \frac{1}{(x-1)(x+1)} > 0$$

$$\frac{(x+1)-x}{x(x-1)(x+1)} > 0$$

$$\frac{1}{x(x-1)(x+1)} > 0$$

Multiply by $x^2(x-1)^2(x+1)^2$:

$$x(x-1)(x+1) > 0$$



The graph is above the x-axis for $-1 < x < 0$, $x > 1$, which is the required solution.

Using **Method 5** (from Example **3**):

$x \neq 0, 1, -1$

Solve $\dfrac{1}{x^2 - x} = \dfrac{1}{x^2 - 1}$: $x^2 - 1 = x^2 - x$

$x = 1$

Critical values are 0, 1, −1; all must be not included (i.e. open circles on number line)

Test $x = -2$:

Is $\dfrac{1}{((-2)^2 - (-2))} > \dfrac{1}{((-2)^2 - 1)}$? No

Test $x = -\frac{1}{2}$:

Is $\dfrac{1}{((-\frac{1}{2})^2 - (-\frac{1}{2}))} > \dfrac{1}{((-\frac{1}{2})^2 - 1)}$? Yes

Test $x = \frac{1}{2}$:

Is $\dfrac{1}{\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\right)} > \dfrac{1}{\left(\left(\frac{1}{2}\right)^2 - 1\right)}$? No

Test $x = 2$:

Is $\dfrac{1}{((2)^2 - (2))} > \dfrac{1}{((2)^2 - 1)}$? Yes

The solution is $-1 < x < 0$, $x > 1$.