PERMUTATIONS

Definition: A permutation is an ordered selection or arrangement of all part of a set of objects

It is very important to remember that permutations are used <u>when the order is important.</u>

Example 7

From a standard pack of 52 playing cards, the set of hearts (♥) are taken. In how many ways:

(a) can the 13 cards be arranged in a row

(b) can 6 of the 13 cards be arranged in a row?

Solution

(a) 13 cards can be arranged in 13! ways.

(b) 6 out of 13 cards can be arranged in $13 \times 12 \times 11 \times 10 \times 9 \times 8 = 1\,235\,520$ ways.

This could be written as $\frac{13!}{(13-6)!} = \frac{13!}{7!} = 1\,235\,520$ ways.

Therefore, the number of arrangements (or permutations) of r objects taken from a group of n objects, which is noted ${}^nP_r$, is:

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

Explanation:

The first place can be filled in n ways, because any one of the n objects can occupy this place. When the first place has been filled in any one of these ways, there remain $(n-1)$ objects, any one of which can occupy the second place. By the fundamental counting principle, each way of filling the first place can be associated with each way of filling the second place, so the number of ways of filling the first two places is $n(n-1)$.

Similarly, after the first two places have been filled in any one of these ways, there remain $(n-2)$ objects, any one of which can occupy the third place. Hence the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing this pattern, the number of ways of filling r places is given by:

$$ n(n-1)(n-2)\ldots(n-r+1) $$

This expression can be simplified as: $\frac{n!}{(n-r)!}$

Example 8

(a) In how many ways can a first, second and third prize be awarded in a class of 10 students?

(b) In how many ways can a Mathematics prize, a Physics prize and a Chemistry prize be awarded in a class of 10 students?

Solution

(a) This question is asking us to find how many different arrangements of three students can be selected from 10 students.

Number of ways $= {}^{10}P_3 = 10 \times 9 \times 8 = 720$

(b) In this question each student could win any or all of the prizes, so permutations are not used.

Number of ways $= 10 \times 10 \times 10 = 1000$

Example 11

In how many ways can the four people, Alexia, Bronwyn, Chanda and Divya, be arranged in a circle?

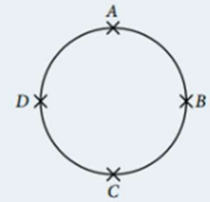
Solution

Method 1

There is no first place to fill, so any one person can be fixed as the 'starting' person (or the 'fixed position') and the other three people arranged around her. This can be done in 3! ways.

Method 2

The diagram at right shows an arrangement of the four people (labelled with the first letter of each name, *A*, *B*, *C*, *D*).



If *A*, *B*, *C*, *D* are kept in the same position relative to each other and then moved all one place clockwise, they still have the same arrangement—the arrangement has not changed.

If *A*, *B*, *C*, *D* were arranged in a straight line and they were all moved along one position to the right (so that the fourth position moved to the first position), then the arrangement would have changed.

A	*B*	*C*	*D*
D	*A*	*B*	*C*
C	*D*	*A*	*B*
B	*C*	*D*	*A*

You should be able to see that for every one arrangement of the group in a circle, there are four different arrangements of the same group in a straight line.

$\therefore$ Number of circular arrangements = Number of linear arrangements ÷ Number of elements

$$= \frac{4!}{4} = 3! = 6$$

Example 12

How many numbers greater than 6000 can be formed using the digits 3, 4, 6, 8, 9 if no digit can be used more than once per number?

Solution

Numbers containing either four digits *or* five digits can be formed.

For four-digit numbers, the first place can be filled in three ways by either 6, 8 or 9.

After the first place is filled (in any one of these three ways), four digits remain, any one of which can occupy second place. Following the multiplication principle, the second place can be filled in four ways, the third place in three ways and the fourth place in two ways:

$\therefore$ Number of arrangements $= 3 \times 4 \times 3 \times 2 = 72$

3	4	3	2

or Number of arrangements $= 3 \times {}^4P_3 = 3 \times \frac{4!}{(4-3)!} = 72$

For five-digit numbers the first place can be filled in five ways, the second in four ways, and so on.

$\therefore$ Number of arrangements $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

5	4	3	2	1

or Number of arrangements $= {}^5P_5 = \frac{5!}{(5-5)!} = 120$

You cannot form a four-digit number and a five-digit number at the same time. These two operations are mutually exclusive.

$\therefore$ Total number of arrangements = 4-digit arrangements + 5-digit arrangements = 72 + 120 = 192