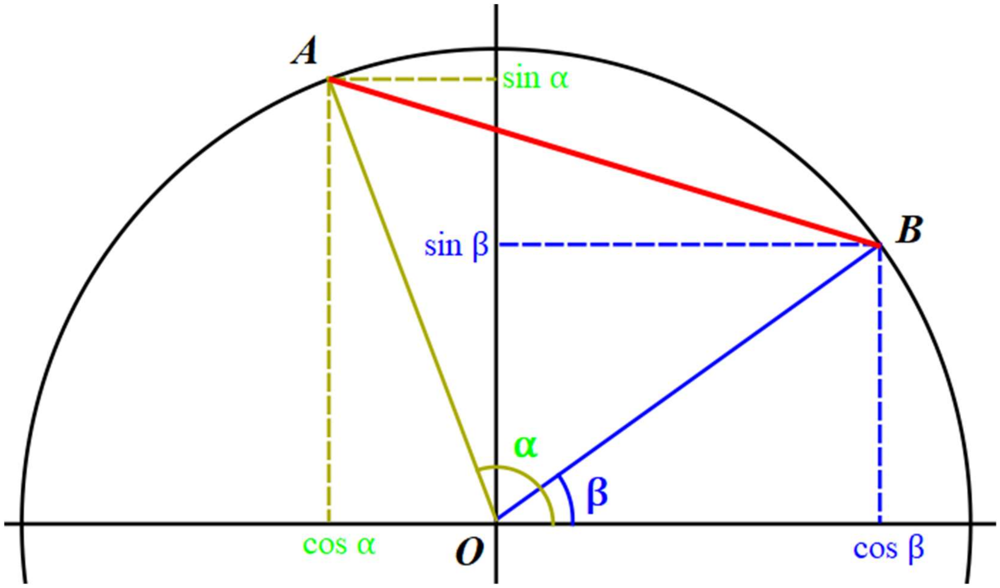
SUM AND DIFFERENCE OF TWO ANGLES
DOUBLE ANGLE FORMULAE



$$\boldsymbol{cos(\alpha - \beta) = cos\,\alpha\,cos\,\beta + sin\,\alpha\,sin\,\beta}$$

Proof: The length AB can be calculated with Pythagoras theorem:

$$AB^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$

$AB^2 = \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$

$AB^2 = \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$

$AB^2 = (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\,(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$AB^2 = 1 + 1 - 2\,(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

Therefore: $AB^2 = 2 - 2\,(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$ Equation (1)

The same length AB can also be calculated applying the cosine rule to triangle OAB.

$$AB^2 = OA^2 + OB^2 - 2 \times OA \times OB\ \cos(\alpha - \beta)$$

but $OA = OB = 1$ so this simplifies as: $AB^2 = 1^2 + 1^2 - 2 \times 1 \times 1\ \cos(\alpha - \beta)$

Therefore $AB^2 = 2 - 2\ \cos(\alpha - \beta)$ Equation (2)

Equalling Equations (1) and (2), we obtain:

$$2 - 2\,(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\ \cos(\alpha - \beta)$$

which simplifies as:

$$\mathbf{cos}(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \mathbf{cos}\,\boldsymbol{\alpha}\,\mathbf{cos}\,\boldsymbol{\beta} + \mathbf{sin}\,\boldsymbol{\alpha}\,\mathbf{sin}\,\boldsymbol{\beta}$$

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$$\boldsymbol{cos(\alpha+\beta)=cos\,\alpha\,cos\,\beta-sin\,\alpha\,sin\,\beta}$$

Proof:

$cos(\alpha+\beta)=cos(\alpha-(-\beta))$

$cos(\alpha+\beta)=cos\,\alpha\,cos(-\beta)+sin\,\alpha\,sin(-\beta)$ using the formula from above

But $cos(-\beta)=cos\,\beta$ and $sin(-\beta)=-\sin\beta$

Therefore $cos(\alpha+\beta)=cos\,\alpha\,cos\,\beta-sin\,\alpha\,sin\,\beta$

Particularly, when $\alpha=\beta$, we obtain: $cos(2\alpha)=cos^2\alpha-sin^2\alpha$

which as $\sin^2\alpha+\cos^2\alpha=1$ can also be written as:

$cos(2\alpha)=2\,cos^2\alpha-1$ or

$cos(2\alpha)=1-2\,sin^2\alpha$

$$\boldsymbol{sin(\alpha+\beta)=sin\,\alpha\,cos\,\beta+cos\,\alpha\,sin\,\beta}$$

Proof:

$sin(\alpha+\beta)=cos[90-(\alpha+\beta)]$

$sin(\alpha+\beta)=cos[90-\alpha-\beta]$

$sin(\alpha+\beta)=cos[(90-\alpha)-\beta]$

Now, as demonstrated above, $cos(\alpha-\beta)=cos\,\alpha\,cos\,\beta+sin\,\alpha\,sin\,\beta$ therefore:

$sin(\alpha+\beta)=cos(90-\alpha)\,cos\,\beta+sin(90-\alpha)\,sin\,\beta$

But $cos(90-\alpha)=sin\,\alpha$ and $sin(90-\alpha)=cos\,\alpha$

therefore: $sin(\alpha+\beta)=sin\,\alpha\,cos\,\beta+cos\,\alpha\,sin\,\beta$

Particularly, when $\alpha=\beta$, we obtain: $sin(2\alpha)=2\,sin\,\alpha\,cos\,\alpha$

$$\boldsymbol{sin(\alpha-\beta)=sin\,\alpha\,cos\,\beta-cos\,\alpha\,sin\,\beta}$$

Proof:

$sin(\alpha-\beta)=\sin[\alpha+(-\beta)]$

Now, as demonstrated above, $sin(\alpha+\beta)=sin\,\alpha\,cos\,\beta+cos\,\alpha\,sin\,\beta$ therefore:

$sin(\alpha-\beta)=sin\,\alpha\,cos(-\beta)+cos\,\alpha\,sin(-\beta)$

But $cos(-\beta)=cos\,\beta$ and $sin(-\beta)=-\sin\beta$

therefore: $sin(\alpha-\beta)=sin\,\alpha\,cos\,\beta-cos\,\alpha\,sin\,\beta$

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$$\boldsymbol{tan(\alpha + \beta) = \frac{tan\,\alpha + tan\,\beta}{1 - tan\,\alpha\, tan\,\beta}}$$

By definition: $tan(\alpha + \beta) = \frac{sin(\alpha+\beta)}{cos(\alpha+\beta)}$

Substituting with the formulas from above, we obtain:

$$tan(\alpha + \beta) = \frac{sin\,\alpha\, cos\,\beta + cos\,\alpha\, sin\,\beta}{cos\,\alpha\, cos\,\beta - sin\,\alpha\, sin\,\beta}$$

We divide both numerator and denominator by $(cos\,\alpha\, cos\,\beta)$

$$tan(\alpha + \beta) = \frac{\frac{sin\,\alpha\, cos\,\beta + cos\,\alpha\, sin\,\beta}{cos\,\alpha\, cos\,\beta}}{1 - \frac{sin\,\alpha\, sin\,\beta}{cos\,\alpha\, cos\,\beta}}$$

$$tan(\alpha + \beta) = \frac{\frac{sin\,\alpha\, cos\,\beta}{cos\,\alpha\, cos\,\beta} + \frac{cos\,\alpha\, sin\,\beta}{cos\,\alpha\, cos\,\beta}}{1 - \frac{sin\,\alpha\, sin\,\beta}{cos\,\alpha\, cos\,\beta}}$$

$$tan(\alpha + \beta) = \frac{\frac{sin\,\alpha}{cos\,\alpha} + \frac{sin\,\beta}{cos\,\beta}}{1 - \frac{sin\,\alpha}{cos\,\alpha}\frac{sin\,\beta}{cos\,\beta}}$$

Therefore: $tan(\alpha + \beta) = \frac{tan\,\alpha + tan\,\beta}{1 - tan\,\alpha\, tan\,\beta}$

Particularly, when $\alpha = \beta$, we obtain: $tan(2\alpha) = \frac{2\,tan\,\alpha}{1 - tan^2\,\alpha}$

$$\boldsymbol{tan(\alpha - \beta) = \frac{tan\,\alpha - tan\,\beta}{1 + tan\,\alpha\, tan\,\beta}}$$

$tan(\alpha - \beta) = tan[\alpha + (-\beta)]$

We use the formula from above

$$tan(\alpha - \beta) = \frac{tan\,\alpha + tan(-\beta)}{1 - tan\,\alpha\ \ tan(-\beta)}$$

But: $tan(-\beta) = -tan\beta$

therefore: $tan(\alpha - \beta) = \frac{tan\,\alpha - tan\,\beta}{1 + tan\,\alpha\ \ tan\,\beta}$

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Example 1

Find the expansion for each expression, simplifying where possible.

(a) $\sin(3x+2y)$ **(b)** $\cos(2\alpha+\beta)$ **(c)** $\tan(A+45°)$

Solution

(a) $\sin(3x+2y) = \sin 3x\cos 2y + \cos 3x\sin 2y$ **(b)** $\cos(2\alpha+\beta) = \cos 2\alpha\cos\beta - \sin 2\alpha\sin\beta$

(c) $\tan(A+45°) = \dfrac{\tan A + \tan 45°}{1 - \tan A\tan 45°} = \dfrac{1+\tan A}{1-\tan A}$

Example 2

Simplify each expression.

(a) $\sin(2\alpha+\beta)\cos\beta - \cos(2\alpha+\beta)\sin\beta$ **(b)** $\cos(2\theta-3\alpha)\cos 2\theta + \sin(2\theta-3\alpha)\sin 2\theta$

Solution

By recognising the form of the equation, the two-angle expansion can be used in reverse:

(a) $\sin(2\alpha+\beta)\cos\beta - \cos(2\alpha+\beta)\sin\beta = \sin[(2\alpha+\beta)-\beta] = \sin 2\alpha$

(b) $\cos(2\theta-3\alpha)\cos 2\theta + \sin(2\theta-3\alpha)\sin 2\theta = \cos[(2\theta-3\alpha)-2\theta]$

$$= \cos(-3\alpha) = \cos 3\alpha$$

Example 3

If θ and ϕ are acute angles and $\sin\theta = \frac{3}{5}$ and $\tan\phi = \frac{24}{7}$, find, without using a calculator, the exact value of the following expressions:

(a) $\sin(\theta+\phi)$ **(b)** $\cos(\theta-\phi)$ **(c)** $\tan(\theta-\phi)$

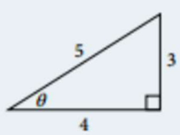
Solution

Draw right-angled triangles for each ratio and use Pythagoras' theorem to find the third side.

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$



$\tan\phi = \frac{24}{7}$

$\sin\phi = \frac{24}{25}$

$\cos\phi = \frac{7}{25}$

(a)
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi = \frac{3}{5}\times\frac{7}{25} + \frac{4}{5}\times\frac{24}{25} = \frac{117}{125}$$

(b)
$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi = \frac{4}{5}\times\frac{7}{25} + \frac{3}{5}\times\frac{24}{25} = \frac{100}{125} = \frac{4}{5}$$

(c)
$$\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi} = \frac{\frac{3}{4}-\frac{24}{7}}{1+\frac{3}{4}\times\frac{24}{7}} = \frac{21-96}{28+72} = -\frac{75}{100} = -\frac{3}{4}$$

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Example 4

If $\tan A = -\frac{3}{4}$, $90° < A < 180°$, and $\cos B = \frac{5}{13}$, $0° < B < 90°$, write the exact value of the following.

(a) $\sin(A - B)$ **(b)** $\cos 2A$ **(c)** $\tan(A + B)$ **(d)** $\sin 2B$

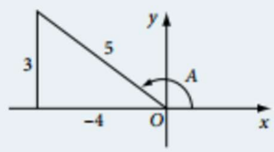
Solution

Draw diagrams to show the given ratio for each angle, then use the diagrams to find the other ratios for the angle.

$\sin A = \frac{3}{5}$

$\cos A = -\frac{4}{5}$

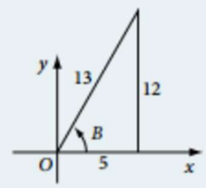
$\tan A = -\frac{3}{4}$



$\sin B = \frac{12}{13}$

$\cos B = \frac{5}{13}$

$\tan B = \frac{12}{5}$



(a) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{63}{65}$

(b) $\cos 2A = \cos^2 A - \sin^2 A$

$= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

(c) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{-\frac{3}{4} + \frac{12}{5}}{1 + \frac{3}{4} \times \frac{12}{5}} = \frac{33}{56}$

(d) $\sin 2B = \sin B \cos B$

$= 2 \times \frac{12}{13} \times \frac{5}{13}$

$= \frac{120}{169}$

Example 5

(a) Prove that $\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = 2 \cot 2\theta$.

(b) Prove that $\frac{\sin 2\alpha + \sin \alpha}{1 + \cos 2\alpha + \cos \alpha} = \tan \alpha$.

Solution

(a) $\text{LHS} = \frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta}$

$= \frac{\cos 3\theta \cos \theta + \sin 3\theta \sin \theta}{\sin \theta \cos \theta}$

$= \frac{\cos(3\theta - \theta)}{\frac{1}{2} \sin 2\theta}$

$= \frac{2 \cos 2\theta}{\sin 2\theta}$

$= 2 \cot 2\theta = \text{RHS}$

(b) **Method 1**

$\text{LHS} = \frac{\sin 2\alpha + \sin \alpha}{1 + \cos 2\alpha + \cos \alpha}$

$= \frac{2 \sin \alpha \cos \alpha + \sin \alpha}{1 + 2\cos^2 \alpha - 1 + \cos \alpha}$

$= \frac{\sin \alpha (2 \cos \alpha + 1)}{\cos \alpha (2 \cos \alpha + 1)}$

$= \frac{\sin \alpha}{\cos \alpha}$ if $2 \cos \alpha + 1 \neq 0$

$= \tan \alpha = \text{RHS}$

Method 2

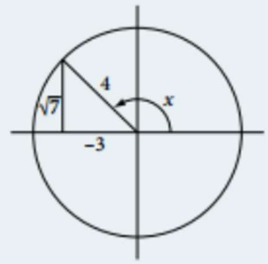
In the denominator, use formula [7] to directly replace $1 + \cos 2\alpha$ with $2\cos^2 \alpha$.

Example 6

If $\cos x = -\frac{3}{4}$ and $\frac{\pi}{2} \le x \le \pi$, find the value of: **(a)** $\sin x$ **(b)** $\sin 2x$

Solution

From the diagram (drawn to show the given ratio for the angle):



(a) $\sin x = \frac{\sqrt{7}}{4}$

(b) $\sin 2x = 2\sin x \cos x$

$$= 2 \times \frac{\sqrt{7}}{4} \times \left(-\frac{3}{4}\right)$$

$$= -\frac{3\sqrt{7}}{8}$$

Example 7

Simplify:

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$ **(b)** $\sin\left(\frac{\pi}{2}-\theta\right) + \sin\left(\frac{\pi}{2}-\phi\right)$ **(c)** $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

Solution

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6} = \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$ (using formula [2] from page 75)

$$= \cos\frac{\pi}{2} = 0$$

(b) $\sin\left(\frac{\pi}{2}-\theta\right) + \sin\left(\frac{\pi}{2}-\phi\right) = \cos\theta + \cos\phi$

(c) $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right) = \frac{1}{2}\times 2\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

$$= \frac{1}{2}\sin 2\left(\frac{\pi}{4}-x\right)$$ (using double-angle formula [8] from page 75)

$$= \frac{1}{2}\sin\left(\frac{\pi}{2}-2x\right) = \frac{1}{2}\cos 2x$$