APPLICATIONS INVOLVING GRAPHING FUNTIONS

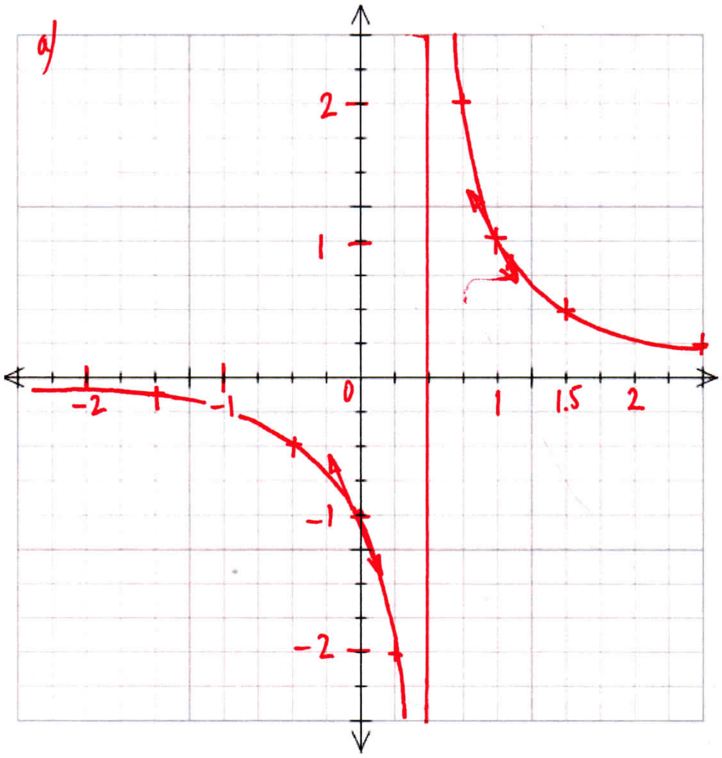
1 **(a)** Sketch the graph of $y = \frac{1}{2x-1}$.

(b) Find the equation of the tangent to the curve at the point where $x = 1$.

(c) Find the equation of the normal to the curve at point where $x = -1$.

(d) Find the coordinates of the point of intersection of the tangent and normal found in parts **(b)** and **(c)**.

a)



b) $f(x) = \frac{1}{2x-1} = (2x-1)^{-1}$

$f'(x) = (-1) \times (2x-1)^{-1-1} \times 2$

$f'(x) = \frac{-2}{(2x-1)^2}$

At $x = 1$ $\quad f'(1) = \frac{-2}{(2\times1-1)^2} = -2$

At $x = 1$ $\quad f(1) = \frac{1}{2-1} = 1$

So the equation of the tangent at $x = 1$ is:

$y - 1 = -2.(x-1)$ or $y = -2x + 3$

c) at $x = -1$ $\quad f'(-1) = \frac{-2}{(2\times(-1)-1)^2} = \frac{-2}{9}$

so the gradient $\perp$ to this tangent is $\frac{9}{2}$.

At $x = -1$ $\quad f(-1) = \frac{1}{2\times(-1)-1} = -\frac{1}{3}$

So the equation of the normal at $x = -1$ is $y + \frac{1}{3} = \frac{9}{2}(x+1)$

or $y = \frac{9}{2}x + \frac{25}{6}$

d) These lines intersect when

$-2x + 3 = \frac{9}{2}x + \frac{25}{6} \iff -\frac{13}{2}x = \frac{7}{6} \iff x = -\frac{14}{78} = -\frac{7}{39}$

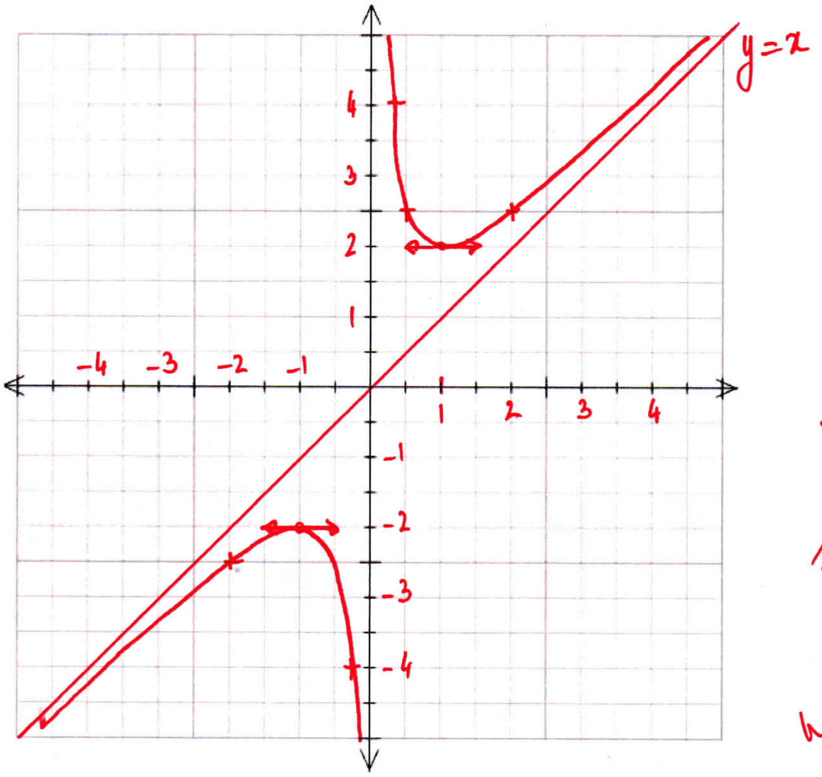
Then $y = -2 \times \left(\frac{-7}{39}\right) + 3 = \frac{131}{39}$ $\quad$ so at $\left(\frac{-7}{39}, \frac{131}{39}\right)$

APPLICATIONS INVOLVING GRAPHING FUNTIONS

2 **(a)** Sketch the curve $y = x + \frac{1}{x}$, showing its asymptotes.

(b) Find the coordinates of the turning points of $y = x + \frac{1}{x}$ and determine their nature.

(c) What is the least value of $x + \frac{1}{x}$ over the domain $x > 0$?



a) When $x \to +\infty$, $\frac{1}{x} \to 0$

so $f(x) \to x$ so $y = x$ is an asymptote.

b) $f'(x) = 1 - \frac{1}{x^2}$

so $f'(x) = 0$ when $1 - \frac{1}{x^2} = 0$

i.e. when $x = \pm 1$

when $x = 1$, $f(1) = 1 + \frac{1}{1} = 2$

when $x = -1$, $f(-1) = -1 + \frac{1}{(-1)} = -2$

So 2 turning points $(1, 2)$ and $(-1, -2)$.

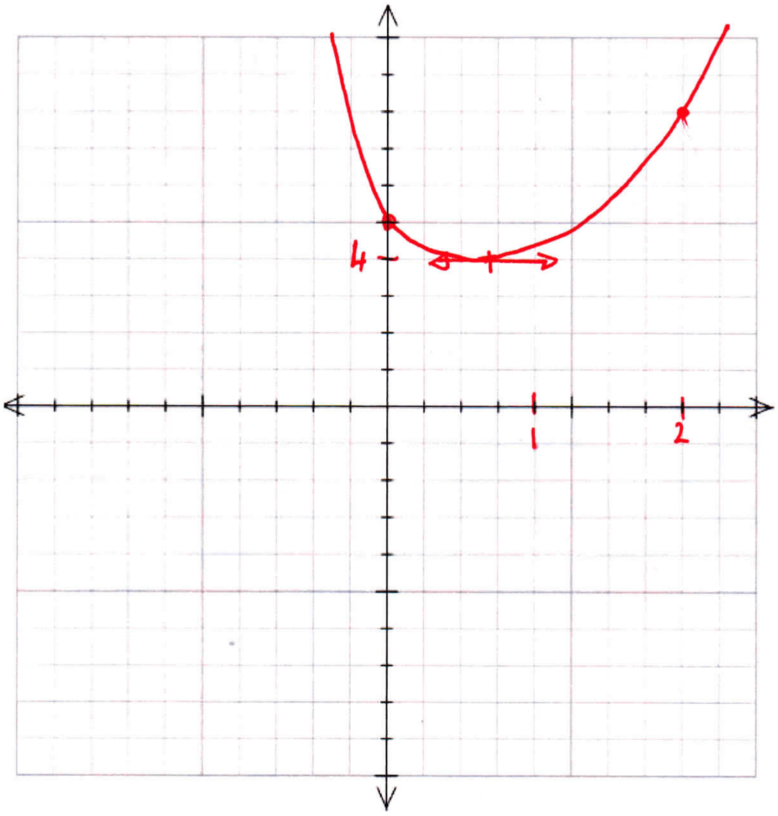
c) The point $(1, 2)$ is a minimum for $x > 0$.

The least value of $f(x)$ when $x > 0$ is 2.

APPLICATIONS INVOLVING GRAPHING FUNTIONS

3 **(a)** Sketch the graph of $f(x) = e^x + 4e^{-x}$. **(b)** For what values of x is $f'(x) > 0$?

(c) What is the minimum value of $f(x)$ and when does it occur?



Note: horizontal and vertical scales are different

$f'(x) = e^x - 4e^{-x}$ so $f'(x) = 0$ when $e^x = 4e^{-x} \Leftrightarrow e^{2x} = 4$

so $2x = \ln 4 = \ln 2^2$ $x = \ln 2 \simeq 0.69$

for $x = \ln 2$, $f(\ln 2) = 2 + 4e^{-\ln 2} = 2 + \frac{4}{2} = 4$

$f'(x) = e^x(1 - 4e^{-2x})$ so $f'(x) > 0$ when $1 - 4e^{-2x} > 0$

$\Leftrightarrow 1 > 4e^{-2x} \Rightarrow \ln \frac{1}{4} > -2x \Rightarrow -\ln 4 > -2x$

$\Rightarrow 2x > \ln 4 \Rightarrow x > \ln 2$

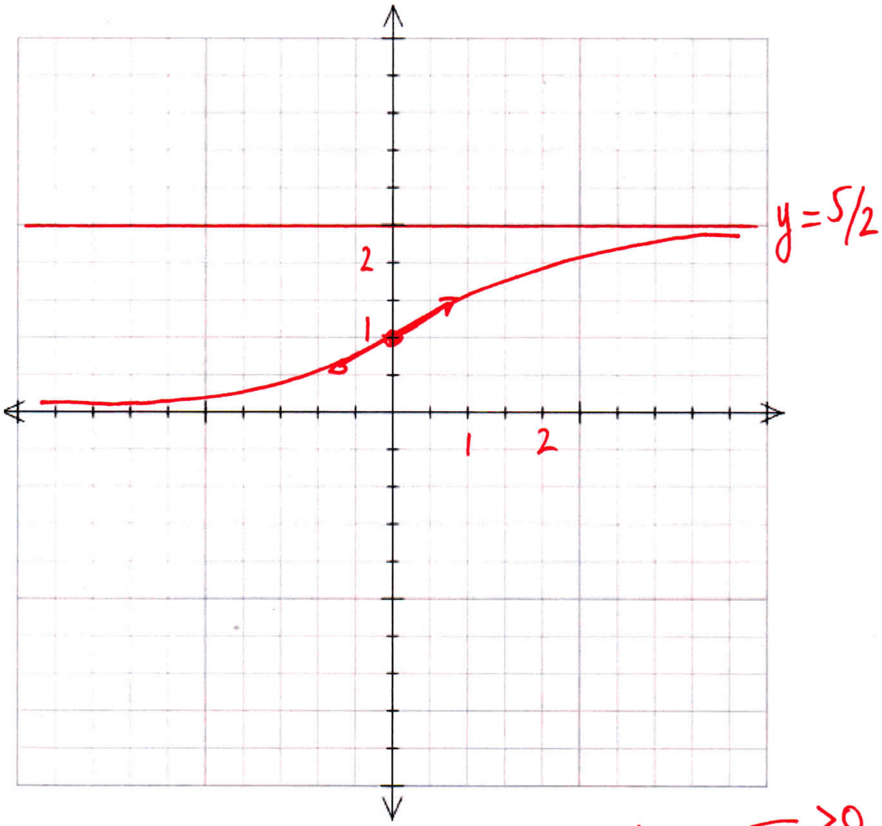
f increasing for $x > \ln 2$, decreasing for $x < \ln 2$

Minimum is at $x = \ln 2$, for which $f(\ln 2) = 4$

APPLICATIONS INVOLVING GRAPHING FUNTIONS

4 **(a)** Sketch the graph of $f(t) = \dfrac{5}{2+3e^{-t}}$, $t \geq 0$. **(b)** Show that $f'(t) > 0$ for all values of t in the domain.

(c) Find $\lim\limits_{t \to \infty} f(t)$. **(d)** What is the range of the function?



$f(0) = \dfrac{5}{2+3} = 1$

b) $f'(t) = \dfrac{5 \times (-1)(-3e^{-t})}{(2+3e^{-t})^2} = \dfrac{15e^{-t}}{(2+3e^{-t})^2}$ >0 >0 so $f'(t) > 0$ on the domain, which is $\mathbb{R}$.

c) When $t \to +\infty$, $e^{-t} \longrightarrow 0$ so $\lim\limits_{t \to +\infty} f(t) = \dfrac{5}{2}$

d) $f(t) = \dfrac{5}{2+3e^{-t}}$. $e^{-t} > 0$ so $2+3e^{-t} > 2$

so $\dfrac{1}{2+3e^{-t}} < \dfrac{1}{2}$

$\dfrac{5}{2+3e^{-t}} < \dfrac{5}{2}$ But $f(t) > 0$ so the range is $(0, \frac{5}{2})$

at 0, $f'(0) = \dfrac{15}{(2+3)^2} = \dfrac{15}{25} = \dfrac{3}{5}$

APPLICATIONS INVOLVING GRAPHING FUNTIONS

5 $f(x)$ is defined by the rule $f(x) = e^{-x}\cos x$ over the domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

(a) Find the values of $f(0)$, $f\left(\frac{\pi}{2}\right)$, $f(\pi)$.

(b) Find $f'(x)$.

(c) Show that $f'(0) = -1$, $f'\left(\frac{3\pi}{4}\right) = 0$ and $f'\left(-\frac{\pi}{4}\right) = 0$.

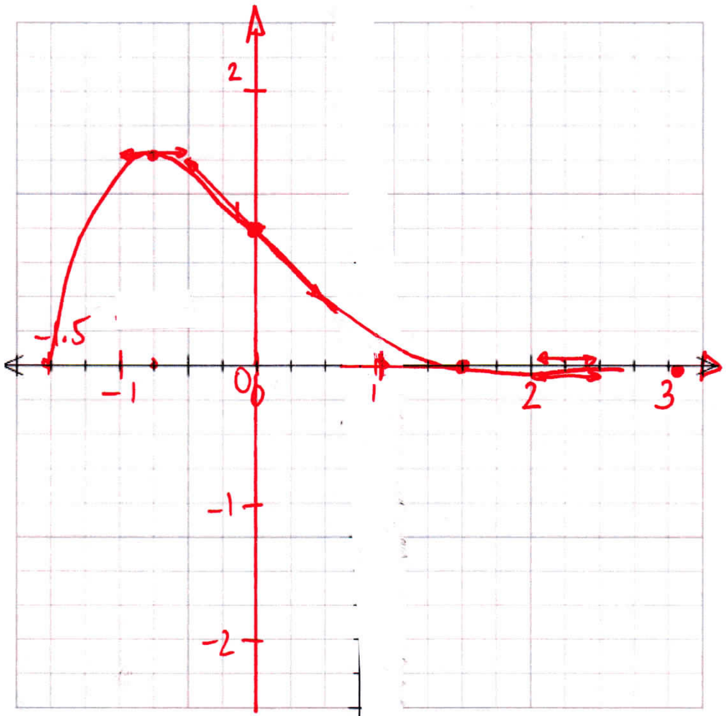
(d) Sketch the graph of $y = f(x)$.

(e) Find the maximum value of $f(x)$ over the domain and the value of x for which it occurs.

a) $f(0) = e^{-0}\cos 0 = 1$

$f\left(\frac{\pi}{2}\right) = 0$

$f(\pi) = e^{-\pi} \times (-1) = -e^{-\pi}$



b) $f'(x) = -e^{-x}\cos x + e^{-x}(-\sin x)$

$f'(x) = -e^{-x}(\cos x + \sin x)$

c) $f'(0) = -e^{-0}(\cos 0 + \sin 0) = -1$

$f'\left(\frac{3\pi}{4}\right) = -e^{-\frac{3\pi}{4}}\left(\underbrace{\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}}_{=0}\right) = 0$

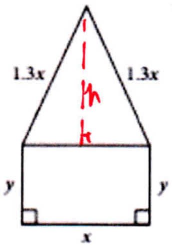
$f'\left(\frac{-\pi}{4}\right) = -e^{\pi/4}\left[\underbrace{\cos\left(\frac{-\pi}{4}\right) + \sin\left(\frac{-\pi}{4}\right)}_{=0}\right] = 0$

d) $f\left(\frac{3\pi}{4}\right) = \frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}$ $\quad f\left(\frac{-\pi}{4}\right) = \frac{e^{\pi/4}}{\sqrt{2}} \sim 1.55$

e) Maximum is for $-\frac{\pi}{4}$, $f\left(\frac{-\pi}{4}\right) = \frac{e^{\pi/4}}{\sqrt{2}}$ so point $\left(\frac{\pi}{4}, \frac{e^{\pi/4}}{\sqrt{2}}\right)$

APPLICATIONS INVOLVING GRAPHING FUNTIONS

9 The diagram consists of a rectangle surmounted by an isosceles triangle with dimensions as shown.



(a) Show that the height of the isosceles triangle is 1.2 x.
(b) Show that the total area of the figure is given by $A = xy + 0.6x^2$.
(c) If the perimeter of the figure is 48 metres, express y in terms of x.
(d) Find the expression for $A(x)$ as a function of x only.
(e) Sketch the graph of $y = A(x)$.
(f) Find the dimensions of the diagram that give a maximum area and state that area.

a) Pythagoras $(1.3x)^2 = h^2 + \left(\frac{x}{2}\right)^2$ so $h^2 = [1.69 - 0.25]\,x^2 = 1.44\,x^2$

$\therefore$ $h = 1.2\,x$

b) $A = xy + \frac{1}{2} \times h \times x = xy + \frac{1.2}{2}x^2 = xy + 0.6\,x^2$

c) $P = 2 \times 1.3\,x + 2y + x = 2y + 3.6\,x$ $\qquad y = \frac{P - 3.6x}{2}$

if $P = 48$, then $y = \frac{48 - 3.6x}{2} = 24 - 1.8x$

d) if $y = 24 - 1.8x$, then $A = x(24 - 1.8x) + 0.6\,x^2$

$A = -1.2x^2 + 24x = 1.2x[-1 + 20x] = 1.2x[20x - 1]$

f) $\frac{dA}{dx} = -2.4x + 24$ A is maximum when $\frac{dA}{dx} = 0$

$\frac{dA}{dx} = 0$ means $-2.4x + 24 = 0 \Rightarrow x = \frac{24}{2.4} = 10$

in that case $y = 24 - 1.8 \times 10 = 6$