THE FIRST AND SECOND DERIVATIVE - CHAPTER REVIEW

1 For the graph of $y = 15x + 12x^2 - 4x^3$ for $-1 \leq x \leq 3$, find the values of x for which:

(a) y increases as x increases

(b) y decreases as x increases

(c) y is a maximum

(d) y is a minimum.

a) y increases as x increases when the curve looks like

i.e, when $\frac{dy}{dx} > 0$

$f'(x) = 15 + 24x - 12x^2$ $\Delta = 24^2 - 4 \times (-12) \times 15 = 36^2$

so 2 roots $x = \frac{-24+36}{2 \times (-12)} = -\frac{1}{2}$ and $x = \frac{-24-36}{2 \times (-12)} = \frac{5}{2}$

the parabola is concave down, there $f(x) > 0$ between the roots.

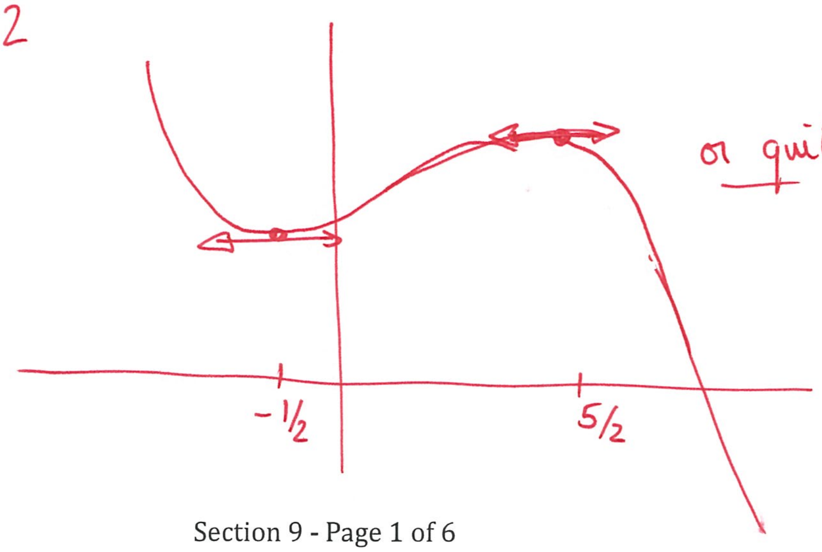
$f'(x) > 0$ when $-\frac{1}{2} < x < \frac{5}{2}$

b) $f'(x) < 0$ when $x < -\frac{1}{2}$ or $x > \frac{5}{2}$

c) $f'(x) = 0$ when $x = -\frac{1}{2}$ or $x = \frac{5}{2}$

At $x = -\frac{1}{2}$ $f'(x) = 0$, with $f'(x) < 0$ for $x < -\frac{1}{2}$ and $f'(x) > 0$ for $x > -\frac{1}{2}$ So $f\left(\frac{-1}{2}\right)$ is a minimum (local)

At $x = \frac{5}{2}$ it's a local maximum.



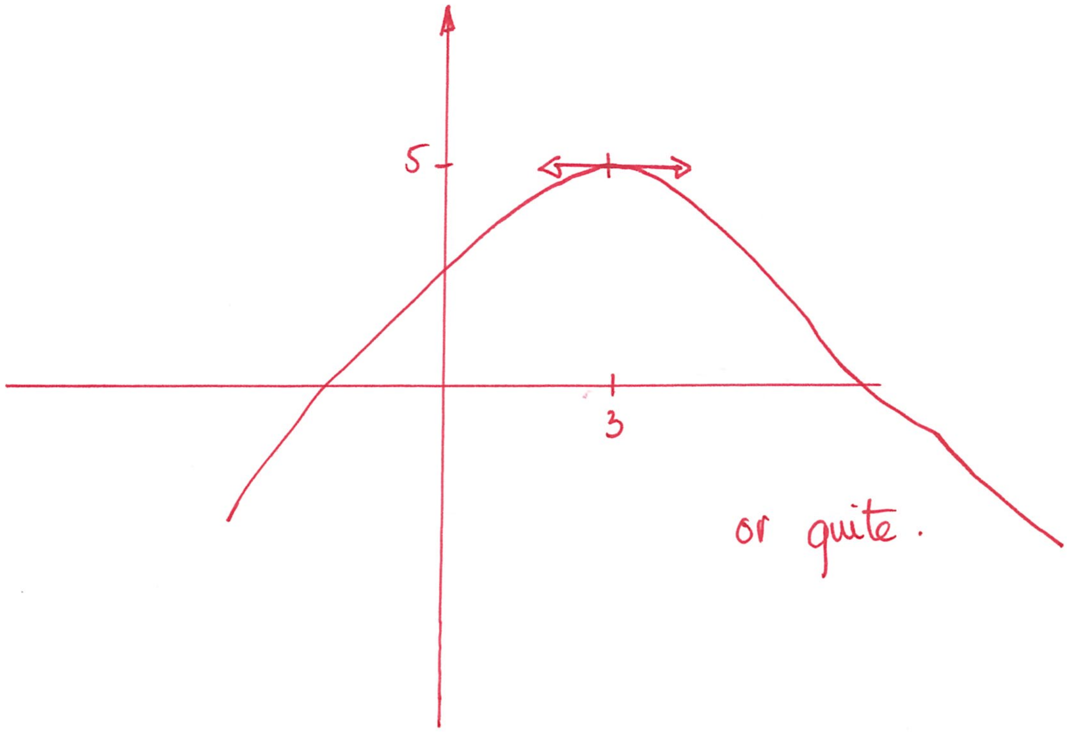
or quite

3 Sketch the graph of $y = f(x)$, given that:

(a) $f(3) = 5$, $f'(3) = 0$, $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$

(b) $f(-1) = 8$, $f'(-1) = 0$, $f(2) = 3$, $f'(2) = 0$, $f'(x) < 0$ for $-1 < x < 2$, and $f'(x) > 0$ for $x < -1$ and for $x > 2$.

a)



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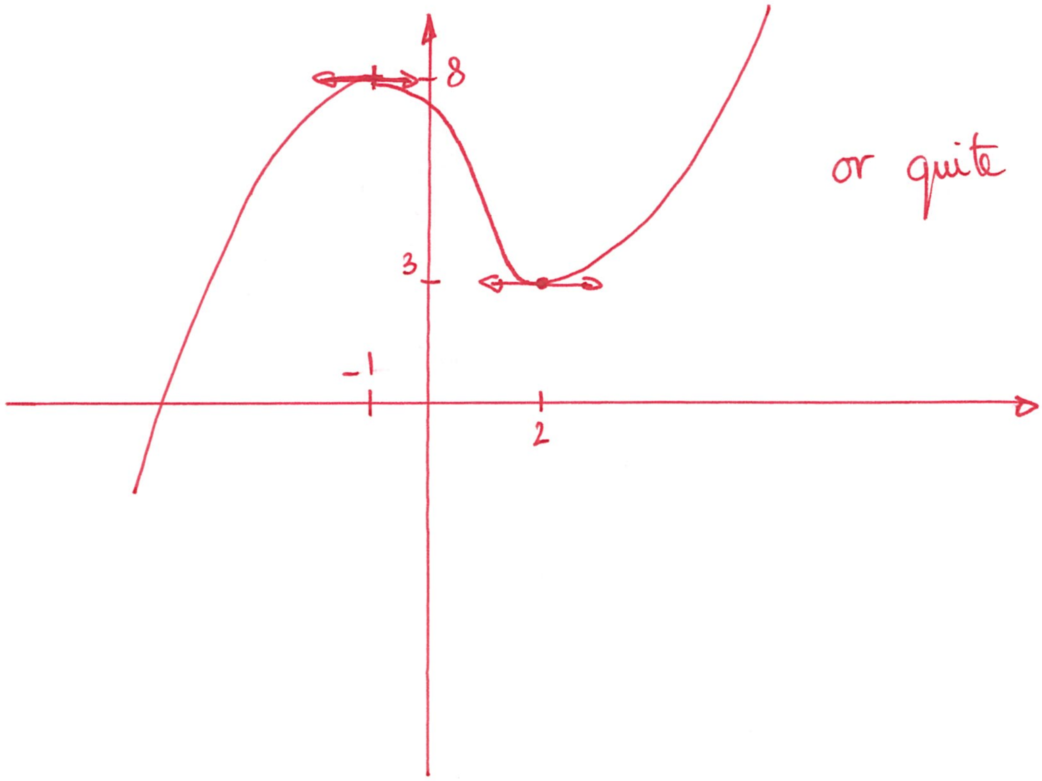
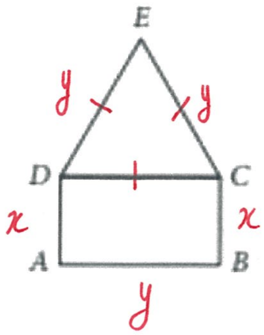
b)



or quite

5 A figure $ABCED$ consists of a rectangle $ABCD$ topped by an equilateral triangle CED as shown in the diagram. If the perimeter of the figure is 45 cm, find the dimensions of the rectangle when the total area is a maximum.



$3y + 2x = 45$ so $y = 15 - \frac{2}{3}x$

$\text{Area} = xy + \frac{1}{2}y \times y \sin 60$

$\text{Area} = xy + \frac{y^2}{2} \times \frac{\sqrt{3}}{2} = xy + \frac{\sqrt{3}\,y^2}{4}$

$\text{Area} = x\left(15 - \frac{2}{3}x\right) + \frac{\sqrt{3}}{4}\left(15 - \frac{2}{3}x\right)^2$

$\text{Area} = x^2\left[-\frac{2}{3} + \frac{4}{9} \times \frac{\sqrt{3}}{4}\right] + 15x - \cancel{2} \times \frac{15}{\cancel{4}}\sqrt{3} \times \frac{\cancel{2}}{3}x$

$f(x) = \text{Area} = x^2\left[\frac{\sqrt{3}}{9} - \frac{2}{3}\right] + 15x\left(1 - \frac{1}{3}\right) = x^2\left[\frac{\sqrt{3}}{9} - \frac{2}{3}\right] + 10x$

This area is maximum when $f'(x) = 0$, i.e.

$f'(x) = 2x\left[\frac{\sqrt{3}}{9} - \frac{2}{3}\right] + 10 = 0$ i.e. $x = \dfrac{-5}{\frac{\sqrt{3} - 6}{9}}$

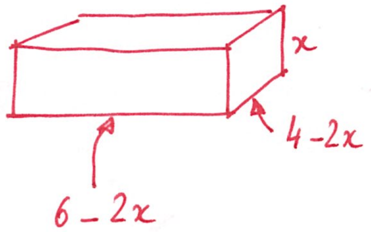
so when $x = \dfrac{45}{6 - \sqrt{3}} = \dfrac{45(6 + \sqrt{3})}{36 - 3} = \dfrac{15(6 + \sqrt{3})}{11}$

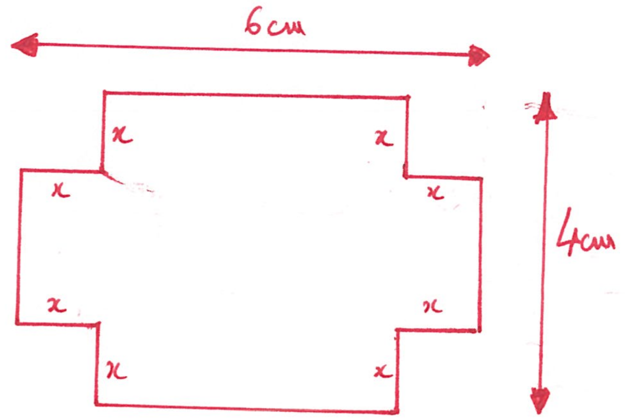
$\therefore\ y = 15 - \frac{2}{3}\left[\frac{15(6 + \sqrt{3})}{11}\right] = 15 - \frac{10(6 + \sqrt{3})}{11}$

$y = \dfrac{105 - 10\sqrt{3}}{11} = \dfrac{5}{11}\left(21 - 2\sqrt{3}\right)$

10 A rectangular sheet of metal measures 6 cm by 4 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the edge length of the squares cut so that the box has a maximum volume.

$$\text{Volume} = (4-2x)(6-2x)\times x$$





$$V = 2(2-x)\times 2(3-x)\times x = 4x(2-x)(3-x)$$

$$V(x) = 4x(6-5x+x^2) = 4x^3-20x^2+24x$$

V(x) is maximum when its derivative is zero

$$V'(x) = 12x^2-40x+24 = 4(3x^2-10x+6)$$

$$\Delta = 100 - 4\times 6\times 3 = 28 = (\sqrt{28})^2 = (2\sqrt{7})^2$$

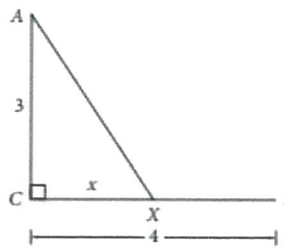
$$x = \frac{10-2\sqrt{7}}{2\times 3} = \frac{5-\sqrt{7}}{3} \quad \text{or} \quad x = \frac{5+\sqrt{7}}{3}$$

we only want the value of x for which V(x) is maximum,

$x = \frac{5+\sqrt{7}}{3}$ is not possible, otherwise the corners would be bigger than the box.

$$\text{So} \quad x = \frac{5-\sqrt{7}}{3} \approx 0.78$$

13 Jack is in the bush at point A, 3 km from the nearest point C, which is at one end of a straight 4 km path CB, as shown in the diagram. Jack wants to get to point B, the other end of the path, as quickly as possible. He can run at a speed of $20\,\text{km}\,\text{h}^{-1}$ along the path CB but only at $10\sqrt{2}\,\text{km}\,\text{h}^{-1}$ in the bush off the path. He runs in a straight line through the bush from A to a point X on the path CB, then along the path from X to B.



(a) Find, in terms of x, the time taken for Jack to go from:
  (i) A to X **(ii)** X to B.

(b) Find, in terms of x, the total time t hours to get from A to B.

(c) Find the position of the point X for which t is a minimum. Find this minimum time.

a) i) $\text{time}(A \to X) = \dfrac{\text{distance}}{\text{speed}} = \dfrac{AX}{10\sqrt{2}} = \dfrac{\sqrt{3^2 + x^2}}{10\sqrt{2}}$

ii) $\text{time}(X \to B) = \dfrac{\text{distance}}{\text{speed}} = \dfrac{XB}{20} = \dfrac{4-x}{20}$

b) $\text{time}(A \to B) = \dfrac{\sqrt{3^2+x^2}}{10\sqrt{2}} + \dfrac{4-x}{20} = f(x)$

c) $f'(x) = \dfrac{1}{10\sqrt{2}} \times \dfrac{1}{\cancel{2}} \times \cancel{2}x \times (9+x^2)^{-1/2} - \dfrac{1}{20} = \dfrac{1}{10}\left[\dfrac{x}{\sqrt{9+x^2}} \times \dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\right]$

$f'(x) = 0$ when $\dfrac{x}{\sqrt{9+x^2}} \times \dfrac{1}{\sqrt{2}} = \dfrac{1}{2} \iff \dfrac{x}{\sqrt{9+x^2}} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

we square both sides.

$\dfrac{x^2}{9+x^2} = \dfrac{1}{2} \Rightarrow 2x^2 = 9 + x^2 \Rightarrow x^2 = 9$

so $x = 3$ km.
(-3 is impossible).

when $x=3$, Minimum time is $\dfrac{\sqrt{9+9}}{10\sqrt{2}} + \dfrac{4-3}{20} = \dfrac{\sqrt{18}}{10\sqrt{2}} + \dfrac{1}{20}$

$= \dfrac{3\sqrt{2}}{10\sqrt{2}} + \dfrac{1}{20} = \dfrac{3}{10} + \dfrac{1}{20} = \dfrac{7}{20}$ hr so 21 minutes

15 **(a)** Find the maximum value of $2xe^{-1.5x}$ and the value for which this function has a maximum value.
(b) If $f(x) = 2xe^{-1.5x}$, find $f(0)$, $f(0.5)$, $f(1)$ and hence graph the function in the domain $0 \le x \le 1$.

a) $f(x) = 2x\, e^{-1.5x}$ $\quad f'(x) = 2\left[x \times (-1.5)\, e^{-1.5x} + 1 \times e^{-1.5x}\right]$

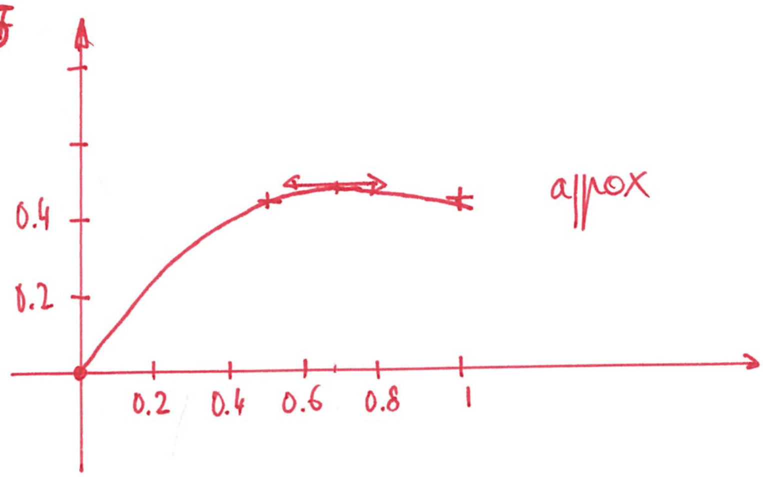
$f'(x) = 2\, e^{-1.5x}\left[-1.5x + 1\right]$

So the function has an extremum at $x = \frac{1}{1.5} = \frac{2}{3}$

this is a maximum as $f'(x) < 0$ when $x < \frac{2}{3}$ and $f'(x) > 0$ when $x > \frac{2}{3}$

b) $f(0) = 0$ $\quad f(0.5) = 2 \times 0.5\, e^{-1.5 \times 0.5} \approx 0.47$

$f(1) = 2 \times 1 \times e^{-1.5 \times 1} \approx 0.4\cancel{5}$



16 If $\theta = \theta_0 e^{-kt}$, show that $\frac{d\theta}{dt} = -k\theta$.

$\theta(t) = \theta_0\, e^{-kt}$ $\quad$ so $\frac{d\theta(t)}{dt} = -k\theta_0\, e^{-kt} = -k \times \theta(t)$

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