THE FACTOR THEOREM

1 If $P(x) = x^4 + x^2 - 2$, indicate whether each statement is correct or incorrect.

(a) $P(1) = 0$ **(b)** $P(2) = 0$ **(c)** $P(-1) = 0$ **(d)** $P(-2) = 0$

$P(1) = 1^4 + 1^2 - 2 = 0$ true

$P(-1) = (-1)^4 + (-1)^2 - 2 = 0$ true

$P(2) = 2^4 + 2^2 - 2 \neq 0$

$P(-2) = (-2)^4 + (-2)^2 - 2$ not equal to 0

2 Use the factor theorem to factorise each polynomial

(a) $x^3 + 4x^2 - 7x - 10$ $P(2) = 0$ so $(x-2)$ is a factor $x - 2 \,\overline{)\, x^3 + 4x^2 - 7x - 10}$ quotient $x^2 + 6x + 5$ $x^3 - 2x^2$ $6x^2 - 7x - 10$ $6x^2 - 12x$ $5x - 10$ $\Delta = 36 - 4 \times 5 = 16 = 4^2$ $x_1 = \frac{-6-4}{2} = -5$ $x_2 = \frac{-6+4}{2} = -1$ So it factorises as $(x-2)(x+5)(x+1)$	**(b)** $x^3 + 2x^2 - 41x - 42$ $P(-1) = 0$ So $(x+1)$ is a factor $P(x) = (x+1)(x^2 + x - 42)$ $\Delta = 1 + 4 \times 42 = 169 = 13^2$ $x_1 = \frac{-1-13}{2} = -7$ $x_2 = \frac{-1+13}{2} = 6$ So it factorises as: $(x+1)(x+7)(x-6)$
(d) $6x^3 - 5x^2 - 12x - 4$ $P(2) = 0$ so $(x-2)$ a factor. $P(x) = (x-2)(6x^2 + 7x + 2)$ $\Delta = 49 - 4 \times 2 \times 6 = 1$ $x_1 = \frac{-7-1}{12} = \frac{-8}{12} = \frac{-2}{3}$ $x_2 = \frac{-7+1}{12} = \frac{-1}{2}$ So it factorises as: $6(x-2)\left(x+\frac{1}{2}\right)\left(x+\frac{2}{3}\right)$	**(f)** $2x^3 + 7x^2 - 10x - 24$ $P(2) = 0$ so $(x-2)$ a factor. $P(x) = (x-2)(2x^2 + 11x + 12)$ $\Delta = 121 - 4 \times 2 \times 12 = 25 = 5^2$ $x_1 = \frac{-11-5}{2} = -8$ $x_2 = \frac{-11+5}{2} = -3$ So it factorises as: $2(x-2)(x+8)(x+3)$

8 If $5x^2 - 6x - 56$ and $3x^2 - 14x + a$ have a common factor $(x + b)$, find the values of a and b.

$(x+b)$ is a factor of $5x^2 - 6x - 56$, therefore $5(-b)^2 - 6(-b) - 56 = 0$

$\Leftrightarrow 5b^2 + 6b - 56 = 0$ $\quad \Delta = 6^2 - 4 \times 5 \times (-56) = 1156 = 34^2$ 2 solutions

$b_1 = \dfrac{-6+34}{2 \times 5} = \dfrac{-28}{10} = -\dfrac{14}{5}$ OR $b_2 = \dfrac{-6-34}{2 \times 5} = \dfrac{-40}{10} = -4$

if $b = -4$, then $(x-4)$ is a factor of $3x^2 - 14x + a$

and therefore $3 \times (+4)^2 - 14 \times 4 + a = 0$ i.e $a = 56 - 48 = 8$

if $b = -14/5$, then $\left(x - \frac{14}{5}\right)$ is a factor of $3x^2 - 14x + a$

and therefore $3 \times \left(\frac{14}{5}\right)^2 - 14 \times \left(-\frac{14}{5}\right) + a = 0$, i.e. $a = \dfrac{-588}{25} - \dfrac{196}{5}$

So $a = -\dfrac{1568}{25}$

In conclusion: if $b = -4$ then $a = 8$; OR if $b = -\frac{14}{5}$ then $a = \dfrac{-1568}{25}$

10 Find the values of a and b that make $2x^3 + ax^2 - 13x + b$ exactly divisible by $x^2 - x - 6$.

We look for the roots of $x^2 - x - 6$

$\Delta = (-1)^2 - 4 \times 1 \times (-6) = 25 = 5^2$

So two roots : $r_1 = \dfrac{1-5}{2} = \dfrac{-4}{2} = -2$ and $r_2 = \dfrac{1+5}{2} = 3$

So $x^2 - x - 6 = (x+2)(x-3)$, and this quadratic divides

$P(x) = 2x^3 + ax^2 - 13x + b$. Therefore $P(-2) = 0$ and $P(3) = 0$.

$P(-2) = 2 \times (-2)^3 + a(-2)^2 - 13 \times (-2) + b = 0$

$\Leftrightarrow -16 + 4a + 26 + b = 0 \Leftrightarrow 4a + b = -10$

$P(3) = 2 \times 3^3 + a \times 3^2 - 13 \times 3 + b = 0$

$\Leftrightarrow 54 + 9a - 39 + b = 0 \Leftrightarrow 9a + b = -15$

2 equations:

$\begin{cases} 4a + b = -10 \\ 9a + b = -15 \end{cases}$

By elimination: $(9a - 4a) = -15 - (-10) = -5$

So $5a = -5$ $\therefore$ $\boxed{a = -1}$

and then $b = -15 - 9 \times (-1)$

$\boxed{b = -6}$

12 Let $P(x) = (x-1)(x+3)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are real numbers. The polynomial $P(x)$ has a factor of $(x+3)$. When $P(x)$ is divided by $(x-1)$ the remainder is 8.

(a) Find the values of a and b.

(b) Find the remainder when $P(x)$ is divided by $(x-1)(x+3)$.

a) $ax+b$ must be $a(x+3)$. so $b = 3a$.

$P(x) = (x-1)(x+3)Q(x) + a(x+3) = (x+3)[(x-1)Q(x) + a]$

$R = P(1) = a + b = 8$ so $a + 3a = 8$ $\boxed{a = 2}$

and then $\boxed{b = 6}$

b) $P(x) = (x+3)[(x-1)Q(x) + 2]$

$P(x) = \underbrace{(x-1)(x+3)Q(x)}_{} + 2x + 6$

this part is divisible by $(x-1)(x+3)$.

So the remainder when $P(x)$ is divided by $(x-1)(x+3)$

is $2x+6$

THE FACTOR THEOREM

13 The polynomial $P(x)$ is given by $P(x) = ax^3 + 12x^2 + cx - 60$, where a and c are constants. The three zeros of $P(x)$ are 2, −3 and β. Find the value of β.

$P(2) = a \times 2^3 + 12 \times 2^2 + c \times 2 - 60 = 0$

i.e $8a + 48 + 2c - 60 = 0$

i.e $4a + c = 6$

$P(-3) = a \times (-3)^3 + 12 \times (-3)^2 + c \times (-3) - 60 = 0$

i.e. $-27a + 108 - 3c - 60 = 0$

$-27a - 3c = -48$

i.e. $9a + c = 16$

By elimination: $5a = 10$ so $\boxed{a = 2}$

and then $c = 6 - 4a = 6 - 8 = -2$ $\boxed{c = -2}$

$P(x) = 2x^3 + 12x^2 - 2x - 60$

$P(x) = 2(x^3 + 6x^2 - x - 30)$

$P(x) = 2(x-2)(x^2 + 8x + 15)$

$P(x) = 2(x-2)(x+3)(x+5)$

So $\boxed{\beta = -5}$