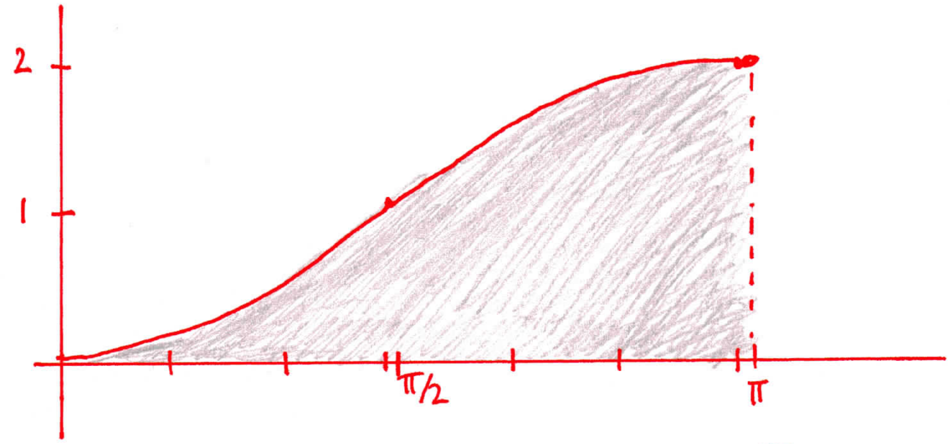
DEFINITE INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

1 Sketch the graph of $f(x) = 1 - \cos x$, $0 \le x \le \pi$. Evaluate $\int_0^{\pi} f(x)dx$ and indicate on the sketch the area represented by this integral. What is the exact value of this area?



$$\int_0^{\pi} (1 - \cos x)\,dx = \left[x - \sin x\right]_0^{\pi}$$

$$= (\pi - \sin\pi) - (0 - \sin 0)$$

$$= \pi - 0$$

$$= \pi \text{ units}^2$$

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2 Evaluate: **(a)** $\int_0^{\pi} \sin x\, dx$ **(b)** $\int_0^{\frac{\pi}{3}} \sec^2 x\, dx$ **(c)** $\int_{\frac{\pi}{3}}^{\pi} \cos \frac{x}{2}\, dx$ **(d)** $\int_0^{\frac{\pi}{2}} \cos x\, dx$

a) $\int_0^{\pi} \sin x\, dx = \left[-\cos x\right]_0^{\pi} = -\cos\pi - (-\cos 0)$

$= -(-1) - (-1) = 1 + 1 = 2$

b) $\int_0^{\pi/3} \sec^2 x\, dx = \left[\tan x\right]_0^{\pi/3} = \tan \pi/3 - \tan 0$

$= \tan \frac{\pi}{3} = \frac{\sin \pi/3}{\cos \pi/3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

c) $\int_{\pi/3}^{\pi} \cos\frac{x}{2}\, dx = \left[2 \sin\frac{x}{2}\right]_{\pi/3}^{\pi} = 2\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right]$

$= 2\left[1 - \frac{1}{2}\right] = 2 \times \frac{1}{2} = 1$

d) $\int_0^{\pi/2} \cos x\, dx = \left[\sin x\right]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$

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(j) $\int_0^{\pi}\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)dx$ **(k)** $\int_0^{\frac{\pi}{3}}\left(3\cos 3x-\frac{\sin 2x}{2}\right)dx$ **(l)** $\int_{-\pi}^{\pi}\left(\frac{\sin x}{2}-\cos x\right)dx$

j) $\int_0^{\pi}\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)dx=\left[4\left(-\cos\frac{x}{4}\right)+4\sin\frac{x}{4}\right]_0^{\pi}=4\left[\sin\frac{x}{4}-\cos\frac{x}{4}\right]_0^{\pi}$

$=4\left\{\left[\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right]-\left[\sin\frac{0}{4}-\cos\frac{0}{4}\right]\right\}=4\left(0-(-1)\right)=4$

k) $\int_0^{\pi/3}\left(3\cos 3x-\frac{\sin 2x}{2}\right)dx=\left[\sin 3x-\frac{1}{2}\frac{(-\cos 2x)}{2}\right]_0^{\pi/3}$

$=\left[\sin 3x+\frac{1}{4}\cos 2x\right]_0^{\pi/3}$

$=\left(\sin\pi+\frac{1}{4}\cos\frac{2\pi}{3}\right)-\left(\sin 0+\frac{1}{4}\cos 0\right)$

$=\frac{1}{4}\times\left(-\frac{1}{2}\right)-\frac{1}{4}=\frac{1}{4}\left(-\frac{1}{2}-1\right)=-\frac{3}{8}$

l) $\int_{-\pi}^{\pi}\left(\frac{\sin x}{2}-\cos x\right)dx=\left[\frac{1}{2}(-\cos x)-\sin x\right]_{-\pi}^{\pi}$

$=\left[-\frac{1}{2}\cos x-\sin x\right]_{-\pi}^{\pi}$

$=\left(-\frac{1}{2}\cos\pi-\sin\pi\right)-\left(-\frac{1}{2}\cos(-\pi)-\sin(-\pi)\right)$

$=\left(-\frac{1}{2}\times(-1)\right)-\left(-\frac{1}{2}\times(-1)\right)$

$=\frac{1}{2}-\frac{1}{2}=0$

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10 Find the points of intersection of the curves $y = \sin\theta$ and $y = \cos\theta$ for $0 \le \theta \le 2\pi$ and calculate the area between the two curves.

The curves intersect when $\sin\theta = \cos\theta \iff \theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$

Between these values $\sin x > \cos x$

So the area is $\displaystyle\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$

$$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

$$= \left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right)$$

$$= \left[-\left(\frac{-\sqrt{2}}{2}\right) - \left(\frac{-\sqrt{2}}{2}\right)\right] - \left[-\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\right]$$

$$= \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] - \left[-\sqrt{2}\right]$$

$$= \sqrt{2} + \sqrt{2}$$

$$= 2\sqrt{2} \text{ units}^2$$

$2\sqrt{2}$ units^2