RESISTANCE PROPORTIONAL TO THE SQUARE OF THE VELOCITY

Use the summary of equations given above and appropriate graphing technology, where necessary, to answer the following questions. Use $g = 10\,\text{m s}^{-2}$.

1 A projectile is fired at an angle of 45° to the horizontal with an initial velocity of $10\sqrt{2}\ \text{m s}^{-1}$.

- **(a)** Write the equation of the trajectory if there is no air resistance.
- **(b)** If air resistance is proportional to the velocity of the projectile, with $k = 0.01$, write the equation of the trajectory in parametric and Cartesian form.
- **(c)** If air resistance is proportional to the square of the velocity of the projectile, with $k = 0.01$, write the equation of the trajectory in parametric form.
- **(d)** Determine, by calculation, the greatest height of attained.
- **(e)** Determine, by calculation, the range of the projectile.
- **(f)** Graph the path of the projectile in each case.
- **(g)** From your graph, determine the greatest height attained in each case.
- **(h)** From your graph, determine the range of the projectile in each case.
- **(i)** Discuss the significance of your answers.

a) $x = u\cos\theta\, t = 10\sqrt{2}\cos 45\, t = 10\sqrt{2}\times\frac{\sqrt{2}}{2}t = 10t$ so $t = x/10$

$y = u\sin\theta\, t - \frac{1}{2}gt^2 = 10\sqrt{2}\sin 45\, t - \frac{1}{2}gt^2 = 10\sqrt{2}\frac{\sqrt{2}}{2}t - \frac{10}{2}t^2 = 10t - 5t^2$
$= 5t[2-t]$

So $y = 5\times\frac{x}{10}\left(2-\frac{x}{10}\right) = \frac{x}{2}\left(2-\frac{x}{10}\right) = x - x^2/20$

b) $x = \frac{u\cos\theta}{k}\left[1-e^{-kt}\right] = \frac{10\sqrt{2}\cos 45}{0.01}\left[1-e^{-0.01t}\right] = 1000\left[1-e^{-0.01t}\right]$ ①

$y = \frac{(g+ku\sin\theta)}{k^2}\left[1-e^{-kt}\right] - \frac{gt}{k} = \frac{(10+0.01\times 10\sqrt{2}\sqrt{2}/2)}{0.01^2}\left[1-e^{-kt}\right] - \frac{10}{0.01}t$

$y = \frac{(10+0.1)}{0.0001}\left[1-e^{-0.01t}\right] - 1000t = 101{,}000\left[1-e^{-0.01t}\right] - 1{,}000t$ ②

From ① $1-e^{-0.01t} = \frac{x}{1000} \Leftrightarrow e^{-0.01t} = 1-\frac{x}{1000}$ so $-0.01t = \ln\left(\frac{1000-x}{1000}\right)$

$\Leftrightarrow t = 100\ln\left(\frac{1000}{1000-x}\right)$

So ② $\Leftrightarrow y = 101{,}000\frac{x}{1000} - 1000\times 100\ln\left[\frac{1000}{1000-x}\right]$

$\Leftrightarrow y = 101x - 100{,}000\ln\left[\frac{1000}{1000-x}\right]$ which is the equation of the trajectory in Cartesian form.

c) $x = \frac{1}{k}\ln\left[1+ku\cos\theta\, t\right] = \frac{1}{0.01}\ln\left[1+0.01\times 10\sqrt{2}\times\frac{\sqrt{2}}{2}t\right] = 100\ln\left[1+\frac{t}{10}\right]$

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$$y = \frac{1}{k}\ln\left|\cos\left(\sqrt{gk}\,t\right) + \sqrt{\frac{k}{g}}\,u\sin\theta\,\sin\left(\sqrt{gk}\,t\right)\right| = \frac{1}{0.01}\ln\left|\cos\sqrt{10\times 0.01}\,t + \sqrt{\frac{0.01}{10}}\times 10\sqrt{2}\,\frac{\sqrt{2}}{2}\sin\left(\sqrt{0.01\times 10}\,t\right)\right|$$

$$y = 100\ln\left|\cos\left(\sqrt{0.1}\,t\right) + \sqrt{0.001}\times 10\sin\left(\sqrt{0.1}\,t\right)\right| = 100\ln\left|\cos\left(\sqrt{0.1}\,t\right) + \sqrt{0.1}\sin\left(\sqrt{0.1}\,t\right)\right|$$

d) For c) $\dot{y} = \dfrac{100}{\cos\left(\sqrt{0.1}\,t\right) + \sqrt{0.1}\sin\left(\sqrt{0.1}\,t\right)} \times \left[\sqrt{0.1}\sin\left(\sqrt{0.1}\,t\right) - \sqrt{0.1}\sqrt{0.1}\cos\left(\sqrt{0.1}\,t\right)\right]$

So $\dot{y} = 0$ when $\sqrt{0.1}\sin\left(\sqrt{0.1}\,t\right) - \sqrt{0.1}\sqrt{0.1}\cos\left(\sqrt{0.1}\,t\right) = 0$

$\iff \sin\left(\sqrt{0.1}\,t\right) = \sqrt{0.1}\cos\left(\sqrt{0.1}\,t\right) \iff \tan\left(\sqrt{0.1}\,t\right) = \sqrt{0.1}$

$\iff \sqrt{0.1}\,t = \tan^{-1}\left(\sqrt{0.1}\right) \iff t = \dfrac{\tan^{-1}\left(\sqrt{0.1}\right)}{\sqrt{0.1}} \simeq 0.97\,s.$

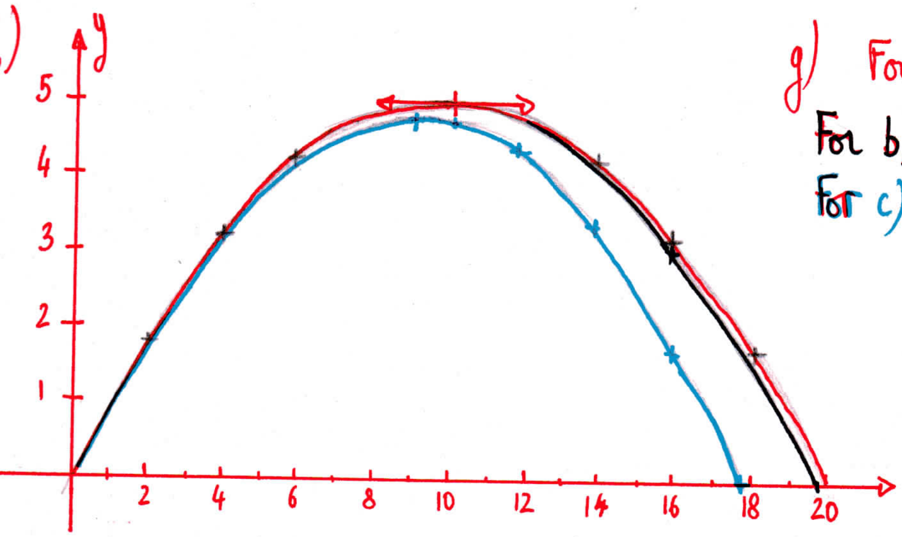
$y = 100\ln\left|\cos\left(\sqrt{0.1}\times 0.97\right) + \sqrt{0.1}\sin\left(\sqrt{0.1}\times 0.97\right)\right| \simeq 4.77\text{ m}$. which is the greatest height

e) Range is when $y = 0$, i.e. $\cos\left(\sqrt{0.1}\,t\right) + \sqrt{0.1}\sin\left(\sqrt{0.1}\,t\right) = 1$

Solving this equation graphically with Desmos gives $t \simeq 1.94\,s$.

Then $x(1.94) = 100\ln\left[1 + \dfrac{1.94}{10}\right] \simeq 17.7$ m which is the range.

f)



g) For a) greatest height is 5 m

For b) greatest height is $\simeq 4.97$ m

For c) greatest height is $\simeq 4.77$ m

h) For a) Range is 20 m For b) it's 19.74 m For c) it's $\simeq 17.7$ m

i) The trajectories with no air resistance and the air resistance proportional to velocity are very similar. For the one with air resistance proportional to the square of velocity, the trajectory is similar until before the greatest height, which is a bit less, and then falls away faster, so range is much less.

2 Repeat question 1 with $k = 0.05$.

a) Same b) $x = \frac{u\cos\theta}{k}\left[1-e^{-kt}\right] = \frac{10\sqrt{2}\times\sqrt{2}/2}{0.05}\left[1-e^{-0.05t}\right]$

So $x = 200\left[1-e^{-0.05t}\right]$ so $-0.05t = \ln\left[\frac{200-x}{200}\right]$ $t = 20\ln\left[\frac{200}{200-x}\right]$

$$y = \frac{(g+ku\sin\theta)}{k^2}\left[1-e^{-kt}\right] - \frac{gt}{k}$$

$$y = \frac{(10+0.05\times10\sqrt{2}\times\sqrt{2}/2)}{(0.05)^2}\left[1-e^{-0.05t}\right] - \frac{10t}{0.05}$$

$$y = \frac{10+0.5}{0.0025}\left[1-e^{-0.05t}\right] - 200t = 4200\left[1-e^{-0.05t}\right] - 200t$$

So $y = 4200\times\frac{x}{200} - 200\times20\ln\left[\frac{200}{200-x}\right]$

$y = 21x - 4{,}000\ln\left(\frac{200}{200-x}\right)$ which is the trajectory in Cartesian coordinates.

c) $x = \frac{1}{k}\ln\left[1+ku\cos\theta\, t\right] = \frac{1}{0.05}\ln\left[1+0.05\times10\sqrt{2}\times\frac{\sqrt{2}}{2}t\right]$

So $x = 20\ln\left[1+0.5t\right]$

$$y = \frac{1}{k}\ln\left|\cos\left(\sqrt{gk}\,t\right) + \sqrt{\frac{k}{g}}\,u\sin\theta\sin\left(\sqrt{gk}\,t\right)\right|$$

$$y = \frac{1}{0.05}\ln\left|\cos\left(\sqrt{10\times0.05}\,t\right) + \sqrt{\frac{0.05}{10}}\times10\sqrt{2}\times\frac{\sqrt{2}}{2}\sin\left(\sqrt{10\times0.05}\,t\right)\right|$$

$$y = 20\ln\left|\cos\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right)\right|$$

d) The greatest height is achieved when $\dot{y} = 0$, i.e.

$$\dot{y} = \frac{20}{\left[\cos\left(\frac{t}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right)\right]}\times\left[\frac{-1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\cos\left(\frac{t}{\sqrt{2}}\right)\right] = \frac{20\left[\frac{1}{\sqrt{2}}\cos\left(\frac{t}{\sqrt{2}}\right)-\sin\left(\frac{t}{\sqrt{2}}\right)\right]}{\sqrt{2}\left[\cos\left(\frac{t}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right)\right]}$$

So $\dot{y}=0$ when $\frac{1}{\sqrt{2}}\cos\left(\frac{t}{\sqrt{2}}\right) - \sin\left(\frac{t}{\sqrt{2}}\right) = 0 \iff \tan\left(\frac{t}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

so $t = \sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \simeq 0.87\,s.$ $\iff \frac{t}{\sqrt{2}} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

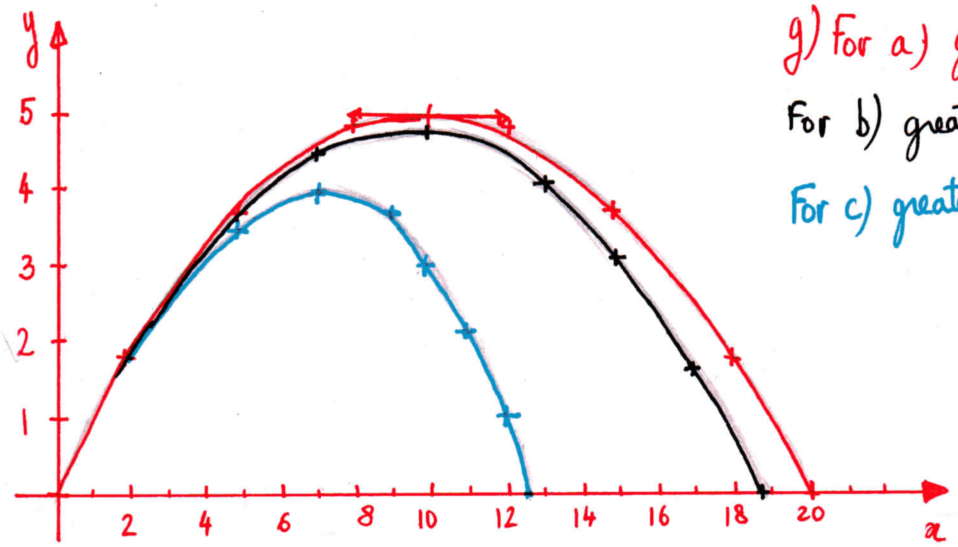
At $t = 0.87\,s$, $y = 20\ln\left|\cos\left(\frac{0.87}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{0.87}{\sqrt{2}}\right)\right| \simeq 4.05\,m$

e) Range is when $y=0$, i.e. $\cos\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right) = 1$

Solving graphically this equation with Desmos gives $t = 1.74\,s$.

For $t = 1.74\,s$, $x = 20\ln\left[1 + 0.5\times 1.74\right] = 12.5\,m$ which is the range.

f)



g) For a) greatest height is 5 m

For b) greatest height is $\simeq 4.8$ m

For c) greatest height is $\simeq 4$ m.

h) For a) Range is 20 m For b) Range is 18.7 approx For c) it's $\simeq 12.5$ m

i) The three trajectories are similar to begin with. The greater the air resistance, the sooner the path drops away, and the steeper the descent.

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3 Repeat question **1** with $k = 0.1$. Compare your answers to questions **1**, **2** and **3**.

a) Same b) $x = \dfrac{u\cos\theta}{k}\left[1 - e^{-kt}\right] = \dfrac{10\sqrt{2} \times \sqrt{2}/2}{0.1}\left[1 - e^{-0.1t}\right]$

So $x = 100\left[1 - e^{-0.1t}\right]$ so $-0.1t = \ln\left(\dfrac{100-x}{100}\right) \Leftrightarrow t = 10\ln\left(\dfrac{100}{100-x}\right)$

$y = \dfrac{(g + ku\sin\theta)}{k^2}\left[1 - e^{-kt}\right] - \dfrac{gt}{k}$

$y = \dfrac{(10 + 0.1 \times 10\sqrt{2} \times \sqrt{2}/2)}{0.1^2}\left[1 - e^{-0.1t}\right] - \dfrac{10t}{0.1}$

$y = 1{,}100\left[1 - e^{-0.1t}\right] - 100t$ in which we substitute $t = 10\ln\left(\dfrac{100}{100-x}\right)$

$y = 1{,}100\left[\dfrac{x}{100}\right] - 100 \times 10\ln\left(\dfrac{100}{100-x}\right)$ So $y = 11x - 1{,}000\ln\left(\dfrac{100}{100-x}\right)$

c) $x = \dfrac{1}{k}\ln\left[1 + ku\cos\theta\, t\right] = \dfrac{1}{0.1}\ln\left[1 + 0.1 \times 10\sqrt{2} \times \dfrac{\sqrt{2}}{2}t\right]$

So $x = 10\ln\left[1 + t\right]$

$y = \dfrac{1}{k}\ln\left|\cos\left(\sqrt{gk}\,t\right) + \sqrt{\dfrac{k}{g}}\,u\sin\theta\sin\left(\sqrt{gk}\,t\right)\right|$

$y = \dfrac{1}{0.1}\ln\left|\cos\left(\sqrt{10 \times 0.1}\,t\right) + \sqrt{\dfrac{0.1}{10}} \times 10\sqrt{2} \times \dfrac{\sqrt{2}}{2}\sin\left(\sqrt{10 \times 0.1}\,t\right)\right|$

$y = 10\ln\left|\cos(t) + \sin(t)\right|$

d) The greatest height is achieved when $\dot{y} = 0$

$\dot{y} = \dfrac{10}{\cos(t) + \sin(t)} \times \left[-\sin(t) + \cos(t)\right] = \dfrac{10\left[\cos(t) - \sin(t)\right]}{\cos(t) + \sin(t)}$

So $\dot{y}=0$ when $\cos t - \sin t = 0 \iff \tan t = 1 \iff t = \pi/4 \simeq 0.79\,s$

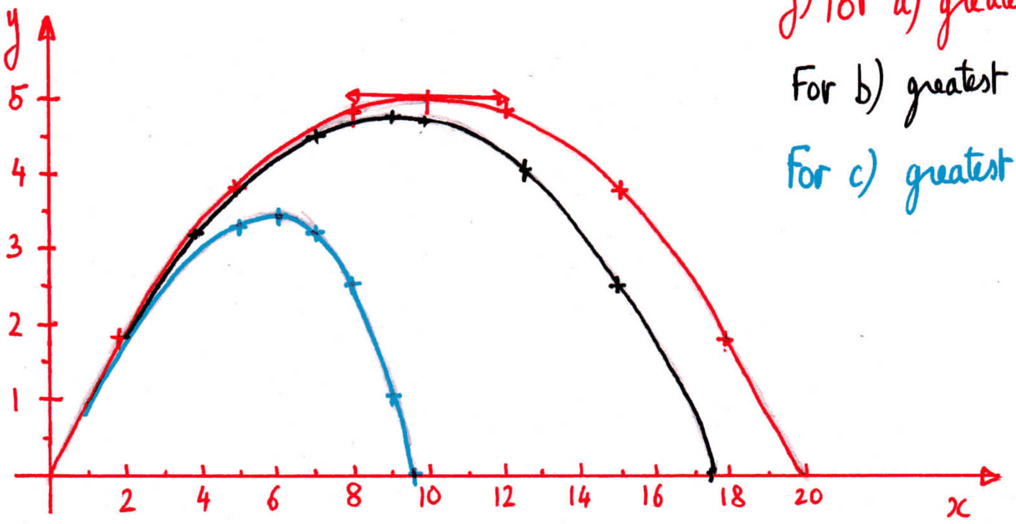
For $t = \pi/4$ $\quad y = 10\ln\left|\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right| = 10\ln\left|\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right| = 10\ln\sqrt{2} \simeq 3.47\,m$

e) Range is when $y = 0$, i.e: $\cos(t) + \sin(t) = 1$

Solving graphically this equation with Desmos gives $t \simeq 1.57\,s$

For $t = 1.57$, $x = 10\ln|1+1.57| \simeq 9.44\,m$ which is the range.

f)



g) For a) greatest height is 5m

For b) greatest height is $\simeq 4.7\,m$

For c) greatest height is $\simeq 3.4\,m$

h) For a) Range is 20 m For b) Range is $\simeq 17.5\,m$ For c) it's $\simeq 9.44\,m$

i) The three trajectories are similar to begin with.
The greater the air resistance, the sooner the path drops away, and the steeper the descent.
As k increases, the greater effect of air resistance reduces the greatest height and the range. When the air resistance is proportional to the square of the velocity, the descent becomes steeper.

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4 A projectile is fired at an angle of 30° to the horizontal with an initial velocity of $6\,m\,s^{-1}$.

(a) Write the equation of the trajectory if there is no air resistance.

(b) If air resistance is proportional to the velocity of the projectile, with $k = 0.02$, write the equation of the trajectory in both parametric form and Cartesian form.

(c) If air resistance is proportional to the square of the velocity of the projectile, with $k = 0.02$, write the equation of the trajectory in parametric form ~~and Cartesian form~~.

(d) Graph the path of the projectile in each case.

(e) From your graph, determine the greatest height attained in each case.

(f) From your graph, determine the range of the projectile in each case.

(g) Discuss the significance of your answers.

a) $x = u\cos\theta\, t = 6\cos 30\, t = 6\times\frac{\sqrt{3}}{2}\, t = 3\sqrt{3}t$ so $t = \frac{x}{3\sqrt{3}}$

$y = u\sin\theta\, t - \frac{1}{2}gt^2 = 6\sin 30\, t - \frac{1}{2}\times 10\, t^2 = \frac{6}{2}t - 5t^2 = 3t - 5t^2$

So $y = 3\times\frac{x}{3\sqrt{3}} - 5\times\left(\frac{x}{3\sqrt{3}}\right)^2 = \frac{x}{\sqrt{3}} - \frac{5x^2}{27}$

b) $x = \frac{u\cos\theta}{k}\left[1 - e^{-kt}\right] = \frac{6\cos 30}{0.02}\left[1 - e^{-0.02t}\right] = 150\sqrt{3}\left[1 - e^{-0.02t}\right]$

$y = \frac{(g + ku\sin\theta)}{k^2}\left[1 - e^{-kt}\right] - \frac{gt}{k} = \frac{(10 + 0.02\times 6\sin 30)}{0.02^2}\left[1 - e^{-0.02t}\right] - \frac{10t}{0.02}$

So $y = \frac{10.06}{0.02^2}\left[1 - e^{-0.02t}\right] - 500t = 25{,}150\left[1 - e^{-0.02t}\right] - 500t$

So $y = 25{,}150\times\frac{x}{150\sqrt{3}} - 500\times\left[50\times\ln\left(\frac{150\sqrt{3}}{150\sqrt{3} - x}\right)\right]$

$y = \frac{503x}{3\sqrt{3}} - 25{,}000\ln\left[\frac{150\sqrt{3}}{150\sqrt{3} - x}\right]$

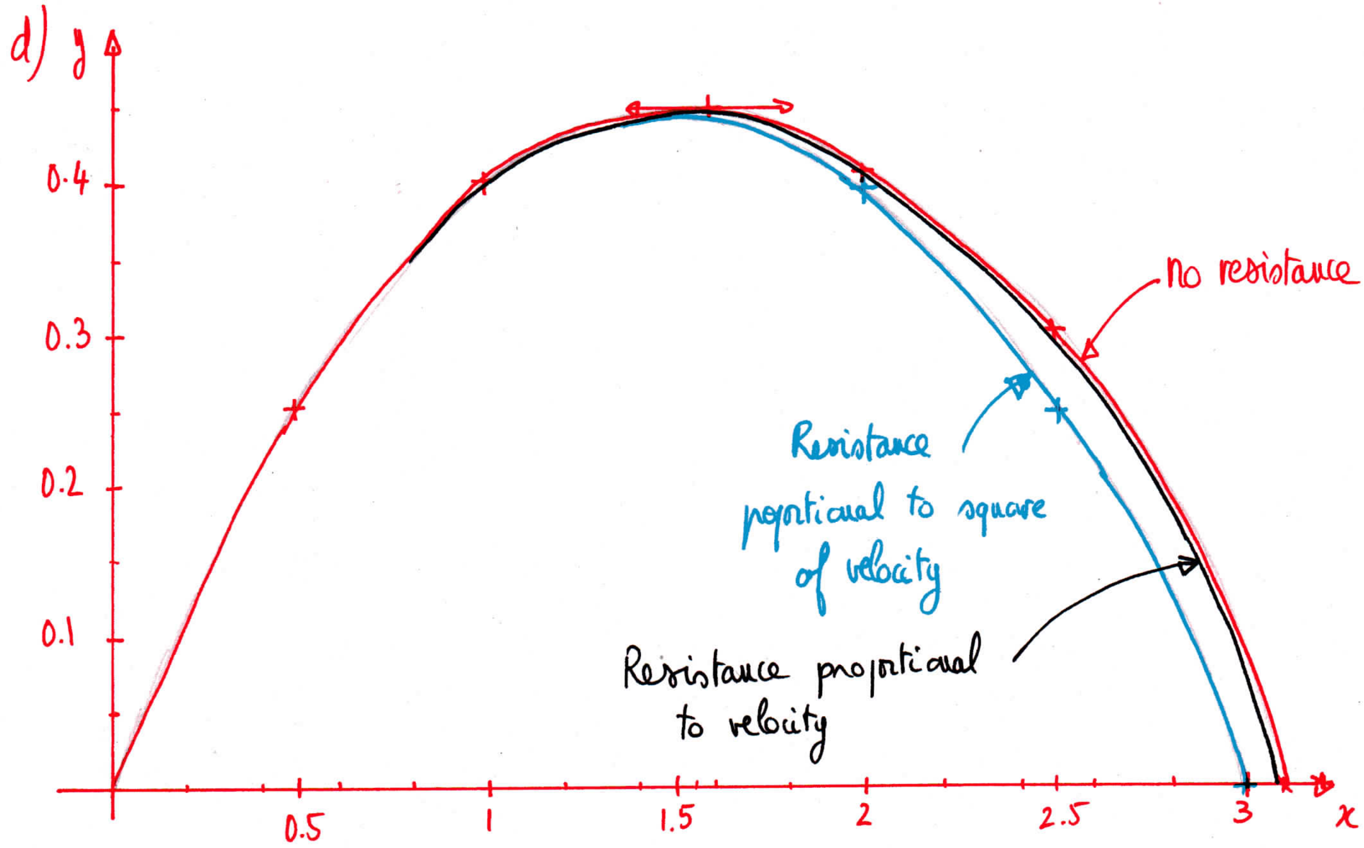
c) $x = \frac{1}{k}\ln\left[1 + ku\cos\theta\, t\right] = \frac{1}{0.02}\ln\left[1 + 0.02\times 6\cos 30\, t\right] = 50\ln\left[1 + 0.06\sqrt{3}\, t\right]$

$y = \frac{1}{k}\ln\left|\cos\left(\sqrt{gk}\,t\right) + \sqrt{\frac{k}{g}}\,u\sin\theta\sin\left(\sqrt{gk}\,t\right)\right|$

$y = \frac{1}{0.02}\ln\left|\cos\left(\sqrt{10\times 0.02}\,t\right) + \sqrt{\frac{0.02}{10}}\times 6\sin 30\sin\left(\sqrt{10\times 0.02}\,t\right)\right|$

$y = 50\ln\left|\cos\left(\sqrt{0.2}\,t\right) + 3\sqrt{0.002}\sin\left(\sqrt{0.2}\,t\right)\right|$

d)



e) Maximum heights reached are very similar, $\simeq 0.45$ m

f) No resistance : Range $\simeq 3.12$ m

Resistance proportional to velocity : Range $\simeq 3.1$ m

Resistance proportional to square of velocity : Range $\simeq 3.0$ m

g) The three trajectories are very similar during the upward motion. The greater the air resistance, the sooner the path drops away, and the steeper the descent.