PROJECTILES AND RESISTED MOTION

Review of projectile motion with no resistance

In *New Senior Mathematics Extension 1 for Years 11 & 12*, Chapter 13, you looked at projectile motion. Unless otherwise stated, any air resistance is ignored when investigating projectile motion and it is assumed that the only force acting on the projectile is the force of gravity. Furthermore, it is assumed that the gravitational force is constant throughout the motion. The following examples **25–27** review projectile motion with no resistance.

Use $\ddot{\underset{\sim}{r}}$ for the acceleration vector, $\dot{\underset{\sim}{r}}$ for the velocity vector and $\underset{\sim}{r}$ for the displacement vector.

Example 25

A golf ball of mass m kg is hit down the middle of a fairway with an initial speed of $25\,\text{m s}^{-1}$ at an angle of projection to the horizontal α, where $\tan\alpha = \frac{4}{3}$.

(a) Taking unit vectors $\underset{\sim}{i}$ horizontally in the direction of motion and $\underset{\sim}{j}$ vertically upward, find an expression for the initial velocity $\underset{\sim}{u}\ \text{m s}^{-1}$ of the ball.

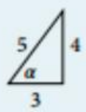
(b) Taking the origin at the point of projection, find an expression for the position vector $\underset{\sim}{r}$ of the ball after t seconds.

(c) Assuming the fairway to be horizontal, find the horizontal distance that the ball travels before hitting the ground. Give your answer correct to one decimal place.

(d) Find the maximum height reached by the ball during its flight. Give your answer correct to one decimal place.

Solution

(a) Draw a diagram.



Express $\underset{\sim}{u}$ in component form: $\underset{\sim}{u} = u\cos\alpha\,\underset{\sim}{i} + u\sin\alpha\,\underset{\sim}{j}$

$$= 25\left(\frac{3}{5}\right)\underset{\sim}{i} + 25\left(\frac{4}{5}\right)\underset{\sim}{j}$$

$$= 15\underset{\sim}{i} + 20\underset{\sim}{j}$$

(b) Determine the equation of motion: If the ball has a mass of m kg, its equation of motion is $-mg\,\underset{\sim}{j} = m\ddot{\underset{\sim}{r}} \quad \therefore\ \ddot{\underset{\sim}{r}} = -g\,\underset{\sim}{j}$

Integrate each component with respect to t to find $\dot{\underset{\sim}{r}}$: $\quad \dot{\underset{\sim}{r}} = -gt\,\underset{\sim}{j} + \underset{\sim}{c}$

Apply the initial condition to find $\underset{\sim}{c}$: $\quad$ At $t = 0$, $\dot{\underset{\sim}{r}} = \underset{\sim}{u}$ so $\underset{\sim}{c} = 15\underset{\sim}{i} + 20\underset{\sim}{j}$

Hence $\dot{\underset{\sim}{r}} = 15\underset{\sim}{i} + (20 - gt)\underset{\sim}{j}$

Integrate each component with respect to t to find $\underset{\sim}{r}$: $\quad \underset{\sim}{r} = 15t\underset{\sim}{i} + \left(20t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

Apply the initial condition to find $\underset{\sim}{d}$: $\quad$ At $t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$ so $\underset{\sim}{d} = \underset{\sim}{0}$

Hence $\underset{\sim}{r} = 15t\underset{\sim}{i} + \left(20t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$

(c) The ball hits the ground when the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ is zero: $20t - 4.9t^2 = 0$

$$t(20 - 4.9t) = 0$$

$$t = 0 \text{ or } t = \frac{20}{4.9} \approx 4.08\,\text{s}$$

The horizontal distance that the ball has covered is the $\underset{\sim}{i}$ component of $\underset{\sim}{r}$:

$$\begin{aligned}\text{Distance} &= 15t \\ &= 15 \times \frac{20}{4.9} \\ &= 61.2\,\text{m}\end{aligned}$$

The ball travels 61.2 m before hitting the ground.

(d) The ball reaches its maximum height when the $\underset{\sim}{j}$ component of $\dot{\underset{\sim}{r}}$ is zero: $20 - 9.8t = 0$

$$t = \frac{20}{9.8}$$

$$t = 2.04\,\text{s}$$

Its height is given by the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$: $h_{max} = h(2.04)$

$$\begin{aligned} &= 2.04\left(20 - 4.9 \times \frac{20}{9.8}\right) \\ &= 2.04 \times 10 \\ &= 20.4\,\text{m}\end{aligned}$$

The ball reaches a maximum height of 20.4 m.

Example 26

A football is kicked towards goal with an initial speed of $26\,\text{m}\,\text{s}^{-1}$ at an angle of projection to the horizontal α, where $\tan\alpha = \frac{5}{12}$.

(a) Find an expression for the initial velocity $\underset{\sim}{u}\,\text{m}\,\text{s}^{-1}$ of the football.

(b) Taking the origin at the point of projection, find an expression for the position vector $\underset{\sim}{r}$ of the football after t seconds.

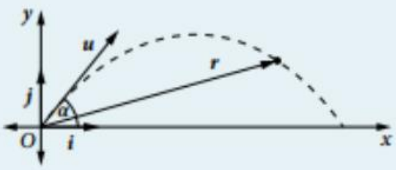
(c) Write the parametric equations of the path of the football and use them to find the Cartesian equation of the path.

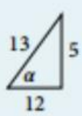
(d) Assuming the ground to be horizontal, find the horizontal distance that the football travels before hitting the ground. Give your answer correct to the nearest metre.

(e) The horizontal crossbar of the goal is 2.44 m above the ground and is 0.1 m wide. If the football is heading towards the goal, which is 40 m from where the football is kicked, will the football pass below the crossbar?

Solution

(a)





$u = 26\,\text{m}\,\text{s}^{-1}$, $\tan\alpha = \frac{5}{12}$ so $\sin\alpha = \frac{5}{\sqrt{5^2 + 12^2}} = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$

$$\begin{aligned}\underset{\sim}{u} &= u\cos\alpha\,\underset{\sim}{i} + u\sin\alpha\,\underset{\sim}{j} \\ &= 26 \times \frac{12}{13}\underset{\sim}{i} + 26 \times \frac{5}{13}\underset{\sim}{j} \\ &= 24\underset{\sim}{i} + 10\underset{\sim}{j}\end{aligned}$$

(b) $\ddot{\underset{\sim}{r}} = -g\underset{\sim}{j}$ as gravity is acting against the motion in the vertical direction.

Integrate with respect to t: $\dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + \underset{\sim}{c}$

$t = 0$, $\underset{\sim}{u} = 24\underset{\sim}{i} + 10\underset{\sim}{j}$: $24\underset{\sim}{i} + 10\underset{\sim}{j} = \underset{\sim}{c}$

$$\dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + 24\underset{\sim}{i} + 10\underset{\sim}{j}$$
$$= 24\underset{\sim}{i} + (10 - gt)\underset{\sim}{j}$$

Integrate with respect to t: $\underset{\sim}{r} = 24t\underset{\sim}{i} + \left(10t - \frac{gt^2}{2}\right)\underset{\sim}{j} + \underset{\sim}{d}$

$t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$: $\underset{\sim}{d} = \underset{\sim}{0}$

$$\underset{\sim}{r} = 24t\underset{\sim}{i} + \left(10t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$$

(c) $x = 24t$, $y = 10t - \frac{1}{2}gt^2$ are the parametric equations of the path.

$t = \frac{x}{24}$, $y = \frac{10x}{24} - \frac{g}{2} \times \left(\frac{x}{24}\right)^2$

If $g = 9.8\,\text{m s}^{-2}$: $y = \frac{10x}{24} - \frac{4.9x^2}{576}$ is the Cartesian equation of the path.

(d) Hits the ground when $y = 0$: $\frac{10x}{24} - \frac{4.9x^2}{576} = 0$

$$x(240 - 4.9x) = 0$$
$$x = 0 \text{ or } x = \frac{240}{4.9} \approx 49\,\text{m}$$

The ball hits the ground 49 metres from where it was kicked.

(e) $x = 40$, $y = \frac{10}{24} \times 40 - \frac{4.9}{576} \times 40^2$

$$\approx 3.06\,\text{m}$$

The ball will pass over the crossbar.

Example 27

A particle is projected from level ground with a velocity of $7\underset{\sim}{i} + 24\underset{\sim}{j}$ m s^{-1}, where $\underset{\sim}{i}$ is horizontal and $\underset{\sim}{j}$ is vertically up. Use $g = 9.8$ m s^{-2}.

(a) Find the initial speed and angle of projection of the particle. Give the angle of projection correct to the nearest tenth of a degree.

(b) Find the time of flight of the particle. Give your answer correct to one decimal place.

(c) Find the horizontal distance travelled by the particle, correct to one decimal place.

(d) Find the maximum height reached by the particle, correct to one decimal place.

(e) Determine whether the particle is ever travelling in a direction perpendicular to its initial velocity.

Solution

(a) $\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$: $\quad |\underset{\sim}{u}| = \sqrt{7^2 + 24^2} = 25 \text{ ms}^{-1}$

Angle of projection θ is $\theta = \tan^{-1}\left(\frac{24}{7}\right) = 73.7°$ to the horizontal

(b) $\ddot{\underset{\sim}{r}} = -g\underset{\sim}{j}$

Integrate with respect to t: $\quad \dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + \underset{\sim}{c}$

$t = 0$, $\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$: $\quad 7\underset{\sim}{i} + 24\underset{\sim}{j} = \underset{\sim}{c}$

$\dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + 7\underset{\sim}{i} + 24\underset{\sim}{j}$

$\quad = 7\underset{\sim}{i} + (24 - gt)\underset{\sim}{j}$

Integrate with respect to t: $\quad \underset{\sim}{r} = 7t\underset{\sim}{i} + \left(24t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\quad \underset{\sim}{d} = \underset{\sim}{0}$

$\underset{\sim}{r} = 7t\underset{\sim}{i} + \left(24t - \frac{1}{2}gt^2\right)\underset{\sim}{j} = 7t\underset{\sim}{i} + \left(24t - 4.9t^2\right)\underset{\sim}{j}$

For the time of flight, find when the vertical component of $\underset{\sim}{r}$ is zero: $24t - 4.9t^2 = 0$

$$t(24 - 4.9t) = 0$$

$$t = 0, t = \frac{24}{4.9} = 4.9 \text{ s}$$

(c) Substitute $t = 4.9$ into the horizontal component of $\underset{\sim}{r}$: Distance $= 7 \times 4.9 = 34.3$ m

(d) The time to the greatest height is half the time of flight: $t = 2.45$

Substitute this value of t into the vertical component of $\underset{\sim}{r}$:
Greatest height $= 24 \times 2.45 - 4.9 \times 2.45^2 \approx 29.4$ m

(e) The direction at any instant is given by the velocity function as it is tangential to the position function at any point in the path.

$\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$, $\dot{\underset{\sim}{r}} = 7\underset{\sim}{i} + (24 - 4.9t)\underset{\sim}{j}$. Find when $\underset{\sim}{u} \bullet \dot{\underset{\sim}{r}} = 0$: $\left(7\underset{\sim}{i} + 24\underset{\sim}{j}\right) \bullet \left(7\underset{\sim}{i} + (24 - 9.8t)\underset{\sim}{j}\right) = 0$

$$49 + 24(24 - 9.8t) = 0$$

$$49 + 576 - 235.2t = 0$$

$$t = \frac{625}{235.2} = 2.66 \text{ s}$$

The particle is travelling in a direction perpendicular to the original direction at 2.66 seconds.

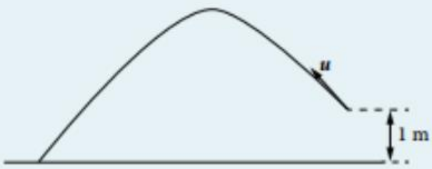
As considered in section 6.5, projectile motion with no air resistance is generally not a realistic model because real projectiles experience resistance due to the medium that they move through. This resistance is a force that can often be approximated as being proportional to the velocity, or proportional to the square of the velocity of the projectile, depending on the conditions of the medium. In this section you will consider resistance proportional to the velocity of the projectile.

Resistance as well as gravity—a vector approach

If there is any resistance other than gravity, it needs to be incorporated into the acceleration vector. The new acceleration vector can then be used to form the other equations of motion.

Example 28

During a game of badminton at the beach, a shuttlecock is hit at a height of 1 m with a velocity of $2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$ m s^{-1}, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively. The acceleration of the shuttlecock due to the combined effect of gravity, air resistance and wind is $2\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k}$ m s^{-2}. Assume the sand to be horizontally flat and take the origin to be at sand level, directly below the point of projection.



(a) Find the time of flight of the shuttlecock. Give your answer correct to two decimal places.

(b) Find where the shuttlecock will land.

(c) Find the maximum height reached by the shuttlecock.

Solution

(a) Define $\ddot{\underset{\sim}{r}}$: $\ddot{\underset{\sim}{r}} = 2\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k}$.

Integrate with respect to t: $\dot{\underset{\sim}{r}} = \int \ddot{\underset{\sim}{r}}\, dt$

$$= \int \left(2\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k}\right) dt$$

$$= 2t\underset{\sim}{i} - t\underset{\sim}{j} - 8t\underset{\sim}{k} + \underset{\sim}{c}$$

$t = 0$, $\dot{\underset{\sim}{r}} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$: $2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k} = \underset{\sim}{c}$

Hence $\dot{\underset{\sim}{r}} = 2t\underset{\sim}{i} - t\underset{\sim}{j} - 8t\underset{\sim}{k} + 2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$

$$= (2 + 2t)\underset{\sim}{i} + (2 - t)\underset{\sim}{j} + (8 - 8t)\underset{\sim}{k}$$

Integrate with respect to t: $\underset{\sim}{r} = \int \left((2 + 2t)\underset{\sim}{i} + (2 - t)\underset{\sim}{j} + (8 - 8t)\underset{\sim}{k}\right) dt$

$$= \left(2t + t^2\right)\underset{\sim}{i} + \left(2t - \frac{t^2}{2}\right)\underset{\sim}{j} + \left(8t - 4t^2\right)\underset{\sim}{k} + \underset{\sim}{d}$$

$t = 0$, $\underset{\sim}{r} = \underset{\sim}{k}$: $\underset{\sim}{k} = \underset{\sim}{d}$

Hence $\underset{\sim}{r} = \left(2t + t^2\right)\underset{\sim}{i} + \left(2t - \frac{t^2}{2}\right)\underset{\sim}{j} + \left(1 + 8t - 4t^2\right)\underset{\sim}{k}$

The shuttlecock hits the sand when the vertical component ($\underset{\sim}{k}$) of the motion is zero: $1 + 8t - 4t^2 = 0$

Rewrite as $4t^2 - 8t - 1 = 0$ and solve: $t = \frac{8 \pm \sqrt{80}}{8} = \frac{2 \pm \sqrt{5}}{2}$

As $t > 0$, the only solution is $t = \frac{2 + \sqrt{5}}{2} \approx 2.12$ s

The time of flight of the shuttlecock is 2.12 seconds.

(b) Find $\underset{\sim}{r}$ when $t = 2.12$ s: $\underset{\sim}{r} = \left(4.24 + 2.12^2\right)\underset{\sim}{i} + \left(4.24 - \frac{2.12^2}{2}\right)\underset{\sim}{j} + 0\underset{\sim}{k}$

$$= 8.73\underset{\sim}{i} + 1.99\underset{\sim}{j}$$

The shuttlecock lands approximately 8.7 m east and 2 m north of its point of projection.

(c) The maximum height is achieved when the $\underset{\sim}{k}$ component of the velocity vector is zero: $8 - 8t = 0$

$$t = 1 \text{ s}$$

Hence the maximum height can be found by substituting $t = 1$ into the $\underset{\sim}{k}$ component of the displacement: $h = 1 + 8(1) - 4(1)^2 = 5$ m.

The shuttlecock reaches a maximum height of 5 metres.

When the resistance to the motion is included in the acceleration vector, the solution to the problem follows the same approach as used earlier. Note that if you are working in three dimensions, coordinates will be needed to locate points in a plane.

Resistance as well as gravity—a Cartesian approach

A particle of mass m is launched at time $t = 0$, from ground level on a flat plane, with an initial velocity of u m s^{-1} at an angle of θ to the horizontal. In addition to gravity, there is an air resistance force, which acts in the opposite direction to the instantaneous direction of motion. The magnitude of this resistance force is directly proportional to the particle's instantaneous speed.

Use standard Cartesian coordinates with the x-axis horizontal and the y-axis vertical. Let the components of the acceleration be $\ddot{x}$ and $\ddot{y}$, so that the components of the velocity are $\dot{x}$ and $\dot{y}$ and so the components of the displacement are x and y.

Initially: $x = 0$, $\dot{x} = u_x = u\cos\theta$, $y = 0$, $\dot{y} = u_y = u\sin\theta$.

Now $m\ddot{x} = -mk\dot{x}$ and $m\ddot{y} = -mg - mk\dot{y}$, where k is a positive constant.

Dividing by m reduces these equations to: $\ddot{x} = -k\dot{x}$, $\ddot{y} = -g - k\dot{y}$

Consider the horizontal motion and let $v_x = \dot{x}$ so that $\ddot{x} = \frac{dv_x}{dt}$: $\quad \frac{dv_x}{dt} = -kv_x$

$$\frac{dv_x}{v_x} = -k\,dt$$

Integrate with respect to t: $\quad \int_{u_x}^{v_x} \frac{dv_x}{v_x} = -k\int_0^t dt$

$$\left[\log_e(v_x)\right]_{u_x}^{v_x} = -kt$$

$$\log_e\left(\frac{v_x}{u_x}\right) = -kt$$

$$v_x = u_x e^{-kt}$$

Now $v_x = \dot{x}$: $\frac{dx}{dt} = u_x e^{-kt}$

Integrate with respect to t: $x = u_x \int_0^t e^{-kt}\,dt$

$$x = -\frac{u_x}{k}\left[e^{-kt}\right]_0^t$$

$$x = -\frac{u_x}{k}\left(e^{-kt} - 1\right)$$

$$x = \frac{u_x}{k}\left(1 - e^{-kt}\right)$$

Consider the vertical motion and let $v_y = \dot{y}$ so that $\ddot{y} = \frac{dv_y}{dt}$: $\quad \frac{dv_y}{dt} = -g - kv_y$

$$\frac{dv_y}{g + kv_y} = -dt$$

Integrate with respect to t: $\quad \int_{u_y}^{v_y} \frac{dv_y}{g + kv_y} = -\int_0^t dt$

$$\left[\frac{1}{k}\log_e\left(g + kv_y\right)\right]_{u_y}^{v_y} = -t$$

$$\log_e\left(g + kv_y\right) - \log_e\left(g + ku_y\right) = -kt$$

$$\log_e\left(\frac{g + kv_y}{g + ku_y}\right) = -kt$$

$$\frac{g+kv_y}{g+ku_y}=e^{-kt}$$

$$kv_y=(g+ku_y)e^{-kt}-g$$

$$v_y=\frac{1}{k}\left((g+ku_y)e^{-kt}-g\right)$$

Now $v_y=\dot{y}$ and $\frac{dy}{dt}=\frac{1}{k}\left((g+ku_y)e^{-kt}-g\right)$.

Integrate with respect to t: $y=\frac{1}{k}\int_0^t\left((g+ku_y)e^{-kt}-g\right)dt$

$$=\frac{1}{k}\left[\frac{g+ku_y}{-k}e^{-kt}-gt\right]_0^t$$

$$=-\frac{g+ku_y}{k^2}e^{-kt}-\frac{gt}{k}+\frac{g+ku_y}{k^2}$$

$$=\frac{g+ku_y}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k}$$

As a result of all of this, you have obtained the parametric equations of the velocity: $\dot{x}=u_xe^{-kt}$, $\dot{y}=\frac{1}{k}\left((g+ku_y)e^{-kt}-g\right)$.

The parametric equations of the path are $x=\frac{u_x}{k}\left(1-e^{-kt}\right)$, $y=\frac{(g+ku_y)}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k}$.

Summary of important results

Projectile motion in a medium whose resistance is proportional to the velocity of the particle:

$t=0, x=0, y=0, \dot{x}=u_x, \dot{y}=u_y$ where $u_x=u\cos\theta$, $u_y=u\sin\theta$, k is the constant of proportionality in the resistance, θ is the angle of projection relative to the horizontal axis, and u is the initial velocity.

$\ddot{x}=-k\dot{x}$, $\ddot{y}=-g-k\dot{y}$

$\dot{x}=u_xe^{-kt}=u\cos\theta e^{-kt}$, $\dot{y}=\frac{1}{k}\left((g+ku_y)e^{-kt}-g\right)=\frac{1}{k}\left((g+ku\sin\theta)e^{-kt}-g\right)=u\sin\theta e^{-kt}-\frac{g}{k}\left(1-e^{-kt}\right)$

$x=\frac{u_x}{k}\left(1-e^{-kt}\right)=\frac{u\cos\theta}{k}\left(1-e^{-kt}\right)$, $y=\frac{(g+ku_y)}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k}=\frac{(g+ku\sin\theta)}{k^2}\left(1-e^{-kt}\right)-\frac{gt}{k}$

With no air resistance, the equations are instead:

Acceleration: $\ddot{x}=0$, $\ddot{y}=-g$

Velocity: $\dot{x}=u\cos\theta$, $\dot{y}=u\sin\theta-gt$

Displacement: $x=u\cos\theta t$, $y=u\sin\theta t-\frac{1}{2}gt^2$

When there is no air resistance, the horizontal velocity is $\dot{x}=u\cos\theta$, whereas with air resistance the horizontal velocity becomes $\dot{x}=u\cos\theta e^{-kt}$, which means the horizontal velocity is reducing over time, decaying exponentially. This means that with air resistance, the particle slows down and will not travel as far.

When there is no air resistance, the vertical velocity is $\dot{y}=u\sin\theta-gt$, whereas with air resistance the vertical velocity becomes $\dot{y}=u\sin\theta e^{-kt}-\frac{g}{k}\left(1-e^{-kt}\right)$. With no air resistance the greatest height is achieved after $\frac{u\sin\theta}{g}$ seconds, but with air resistance it is after $\frac{1}{k}\log_e\left|\frac{ku\sin\theta}{g}+1\right|$ seconds.

With no air resistance, the greatest height attained is $\frac{u^2\sin^2\theta}{2g}$ m; with air resistance it is $\frac{u\sin\theta}{k}-\frac{g}{k^2}\log_e\left|1+\frac{ku\sin\theta}{g}\right|$, which is always less (for sensible positive values of the constants).

This air resistance behaviour is best considered using a numerical example.

Example 29

A particle is projected from a point on the horizontal plane with an initial velocity given by the components $u_x = 3\,\text{m s}^{-1}$, $u_y = 8\,\text{m s}^{-1}$. Use $g = 9.8\,\text{m s}^{-2}$.

(a) If the only resistance to the motion is gravity, find the parametric equations of the trajectory of the particle. When does the particle hit the ground?

The particle is projected again into a medium which resists the motion, where the resistance to the motion is directly proportional to the velocity of the particle. Let the constant of proportionality be k.

The parametric equations of the trajectory are given as $x = \frac{u_x}{k}\left(1 - e^{-kt}\right)$, $y = \frac{\left(g + ku_y\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$.

(b) Find the parametric equations of the trajectory when **(i)** $k = 0.5$ **(ii)** $k = 0.1$.

(c) Using technology, draw on the same set of axes the graphs of the three trajectories in parts **(a)** and **(b)**.

(d) Discuss what your graphs in part **(c)** tell you about the motion in each case.

Solution

(a) No air resistance: $u_x = 3$, $u_y = 8$, $g = 9.8$.

Initial equations of motion:	$\ddot{x} = 0$	$\ddot{y} = -g = -9.8$
Integrate with respect to t:	$\dot{x} = C_1$	$\dot{y} = -9.8t + C_2$
$t = 0, u_x = 3, u_y = 8$:	$\dot{x} = 3$	$\dot{y} = 8 - 9.8t$
Integrate with respect to t:	$x = 3t + C_3$	$y = 8t - 4.9t^2 + C_4$
$t = 0, x = 0, y = 0$:	$x = 3t$	$y = 8t - 4.9t^2$
Hits the ground when $y = 0$:	$8t - 4.9t^2 = 0$	

$$t = \frac{8}{4.9}\text{ s}$$

(b) **(i)** $k = \frac{1}{2}$: $x = \frac{3}{0.5}\left(1 - e^{-\frac{t}{2}}\right)$ $\qquad y = \frac{9.8 + 4}{0.5^2}\left(1 - e^{-\frac{t}{2}}\right) - \frac{9.8t}{0.5}$

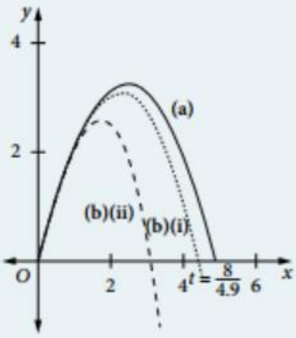
$$x = 6\left(1 - e^{-\frac{t}{2}}\right) \qquad y = 55.2\left(1 - e^{-\frac{t}{2}}\right) - 19.6t$$

(ii) $k = 0.1$: $x = \frac{3}{0.1}\left(1 - e^{-\frac{t}{10}}\right)$ $\qquad y = \frac{9.8 + 4}{0.1^2}\left(1 - e^{-\frac{t}{10}}\right) - \frac{9.8t}{0.1}$

$$x = 30\left(1 - e^{-\frac{t}{10}}\right) \qquad y = 1380\left(1 - e^{-\frac{t}{10}}\right) - 98t$$

(c) In the diagram, the solid line represents part **(a)**, the dotted line represents part **(b)(i)** and the dashed line represents part **(b)(ii)**. Each line has been completed for $t = \frac{8}{4.9}$ seconds, the time taken for the particle without air resistance to return to the ground.

Two of the lines show $y < 0$, but this is just so the line lengths can be compared for the largest value of t in part **(a)**. After each particle hits the ground, it stops, so the parts of the paths below zero do not actually exist.



(d) At the beginning of the motion the paths are similar. The trajectory in part **(a)** is symmetrical about its greatest height.

The larger the air resistance, the sooner the trajectory falls below the path in part **(a)**. The particles in part **(b)** hit the ground before the particle in part **(a)**.

The larger the air resistance, the shorter the range.

The larger the air resistance, the lower the greatest height and the steeper the fall after the particle reaches its greatest height.

Example 30

A projectile is fired from the origin O with an initial velocity u m s^{-1} at an angle θ to the horizontal in a medium whose resistance is proportional to the velocity.

The parametric equations of the trajectory are $x = \dfrac{u\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \dfrac{(10+ku\sin\theta)}{k^2}\left(1-e^{-kt}\right)-\dfrac{10t}{k}$,

where k is the constant of proportionality of the resistance.

The projectile is fired at an angle of 60° to the horizontal with an initial velocity of $10\sqrt{3}$ m s^{-1}, $k = 0.4$.

(a) Find when the projectile reaches its greatest height, correct to two decimal places.

(b) Find the greatest height that is reached, correct to two decimal places.

(c) Show that the projectile hits the ground when $t \approx 2.6$ s **(i)** graphically **(ii)** by substitution.

(d) Find the horizontal range of the projectile.

(e) Graph the path of the projectile.

Solution

Write the parametric equations of the trajectory using the information provided.

$$x = \frac{10\sqrt{3}\times\frac{1}{2}}{0.4}\left(1-e^{-0.4t}\right) \qquad y = \frac{10+0.4\times10\sqrt{3}\times\frac{\sqrt{3}}{2}}{0.4^2}\left(1-e^{-0.4t}\right)-\frac{10t}{0.4}$$

$$= 12.5\sqrt{3}\left(1-e^{-0.4t}\right) \qquad = 100\left(1-e^{-0.4t}\right)-25t$$

(a) Greatest height when $\dot{y} = 0$: $y = 100\left(1-e^{-0.4t}\right)-25t$

$$\dot{y} = 40e^{-0.4t}-25$$

$$40e^{-0.4t}-25 = 0$$

$$e^{-0.4t} = \frac{5}{8}$$

$$-0.4t = \log_e\left(\frac{5}{8}\right)$$

$$t = 2.5\log_e 1.6$$

$$t = 1.18\text{ s}$$

(b) $t = 1.18$: $y = 100\left(1-e^{-0.4\times1.18}\right)-25\times1.18$

$$= 8.12\text{ m}$$

The greatest height is 8.12 metres.

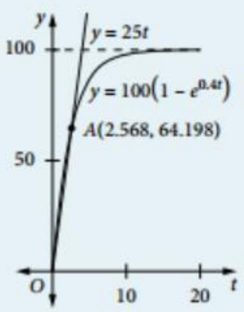
(c) **(i)** Hits the ground when $y = 0$: $\quad y = 100\left(1-e^{-0.4t}\right)-25t$

$$100\left(1-e^{-0.4t}\right)-25t = 0$$

$$100\left(1-e^{-0.4t}\right) = 25t$$

Hence graph $y = 100\left(1-e^{-0.4t}\right)$ and $y = 25t$.

It hits the ground at $t = 2.6$ seconds.



(ii) $t = 2.6$ s, $y = 100\left(1-e^{-0.4t}\right)-25t$

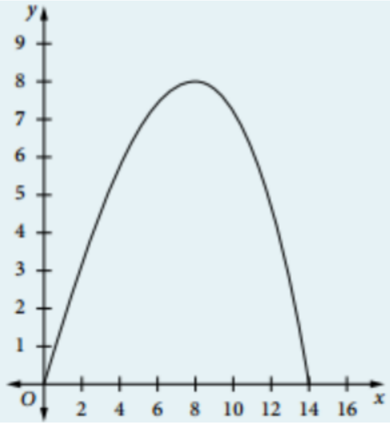
$$= 100\left(1-e^{-1.04}\right)-25\times2.6$$

$$= -0.345 \approx 0$$

It hits the ground at approximately $t = 2.6$ seconds.

(d) $t = 2.6,\ x = 12.5\sqrt{3}\left(1 - e^{-0.4\times 2.6}\right)$
$= 12.5\sqrt{3}\left(1 - e^{-1.04}\right)$
$= 13.998 \approx 14.0\,\text{m}$

(e)



Using terminal velocity

Given the terminal velocity $v_T = \frac{g}{k}$, consider what this means in the equations involving air resistance. Remember, the terminal velocity for a projectile falling vertically downwards is the velocity at which the drag force (air resistance) balances the gravitational force.

Now $v_T = \frac{g}{k}$, $k = \frac{g}{v_T}$ or $\frac{1}{k} = \frac{v_T}{g}$ can be substituted in each of the previous equations for air resistance motion.

Horizontally: $\ddot{x} = -k\dot{x} = -\frac{g\dot{x}}{v_T}$, $\dot{x} = u_x e^{-kt} = u_x e^{\frac{-gt}{v_T}}$, $x = \frac{u_x}{k}\left(1 - e^{-kt}\right) = \frac{u_x v_T}{g}\left(1 - e^{\frac{-gt}{v_T}}\right)$.

When there is no air resistance, the horizontal velocity is $\dot{x} = u_x$, whereas with air resistance the horizontal velocity has become $\dot{x} = u_x e^{\frac{-gt}{v_T}}$, which is reducing over time, decaying exponentially.

Vertically: $\ddot{y} = -g - k\dot{y} = -g\left(1 + \frac{\dot{y}}{v_T}\right)$,

$\dot{y} = \frac{1}{k}\left((g + ku_y)e^{-kt} - g\right) = \left(v_T + u_y\right)e^{\frac{-gt}{v_T}} - v_T = u_y e^{\frac{-gt}{v_T}} - v_T\left(1 - e^{\frac{-gt}{v_T}}\right)$,

$y = \frac{g + ku_y}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k} = \frac{v_T}{g}\left(v_T + u_y\right)\left(1 - e^{\frac{-gt}{v_T}}\right) - v_T t$.

When there is no air resistance, the vertical velocity is $\dot{y} = u_y - gt$, whereas with air resistance the vertical velocity has become $\dot{y} = u_y e^{\frac{-gt}{v_T}} - v_T\left(1 - e^{\frac{-gt}{v_T}}\right)$.

Consider some values for t: $t = \frac{v_T}{g}$, $\dot{x} = u_x e^{-1} \approx 0.37u_x$, $\dot{y} = u_y e^{-1} - v_T\left(1 - e^{-1}\right) \approx 0.37u_y - 0.63v_T$

$t = \frac{2v_T}{g}$, $\dot{x} = u_x e^{-2} \approx 0.14u_x$, $\dot{y} = u_y e^{-2} - v_T\left(1 - e^{-2}\right) \approx 0.14u_y - 0.86v_T$

$t = \frac{3v_T}{g}$, $\dot{x} = u_x e^{-3} \approx 0.05u_x$, $\dot{y} = u_y e^{-3} - v_T\left(1 - e^{-3}\right) \approx 0.05u_y - 0.95v_T$

Because $v_T < 0$, as it is downwards, therefore as t increases above $t = \frac{v_T}{g}$, the horizontal velocity decreases exponentially and becomes negligible, so that the particle appears to be falling vertically.

When $t < \frac{v_T}{g}$, the equations of motion with air resistance approximate the equations of motion where air resistance is ignored.