SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

The previous section investigated techniques used to develop a graphical and/or numerical representation of the solution to a differential equation. However, the most convenient solution (when it is available) is an expression for the dependent variable as an explicit function of the independent variable or as an implicit relation between the dependent and independent variables. This section covers a number of powerful techniques capable of determining these analytical solutions.

Solving $\frac{dy}{dx} = f(x)$ given $y(a) = y_a$

Begin with the directly integrable case $\frac{dy}{dx} = f(x)$ where y is an unknown function of x and $f(x)$ is a given function of x. Wherever $f(x)$ is a continuous function over an interval, $y(x)$ can be determined by using an appropriate integration technique.

Solving a directly integrable first-order differential equation

Example 12

A one-parameter family of curves $f(x, y) = c$ has the property that the gradient of any member of the family at a point is three times the square of the x-coordinate at the point.

(a) What is the equation of the family of functions?

(b) Plot a few members of the family.

(c) What is the equation of the particular member of the family that passes through the point (1, 6)?

Solution

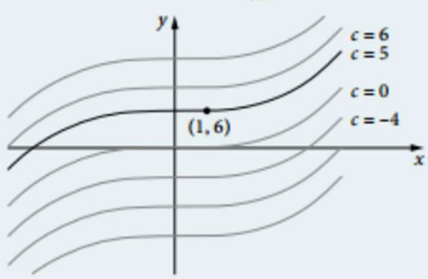
(a) Possible gradient function: $\frac{dy}{dx} = 3x^2$.

Both sides of the model are integrated with respect to the independent variable: $\int \frac{dy}{dx}dx = \int 3x^2 dx$

Equation of the family of functions: $y = x^3 + c$

Note: By the chain rule, $\int \frac{dy}{dx}dx = \int dy$. A constant of integration is required when integrating.

(b)



(c) The coordinates of the given point (1, 6) are substituted into the equation of the family of functions:

$6 = 1^3 + c$

$c = 5$

Solution: $y = x^3 + 5$

Example 13

The slope of the tangent line to an unknown curve is $\frac{1}{1+x^2}$. The y-intercept of the curve is (0, 1). Find the equation of the curve.

Solution

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\int \frac{dy}{dx} dx = \int \frac{dx}{1+x^2}$$

$$y = \tan^{-1} x + c$$

$$y(1) = 0 : 1 = \tan^{-1} 0 + c$$

$$c = 1$$

$$\therefore y = \tan^{-1} x + 1$$

Finding the particular solution of a directly integrable problem $\frac{dy}{dx} = f(x), y(a) = y_a$ involves the following two-step procedure.

1 Integrate with respect to the independent variable.

That is, $y = F(x) + c$ where $F(x) = \int f(x)dx$ and c is the constant of integration.

2 Use the initial condition to solve the constant of integration.

That is, $(x, y) = (a, y_a) \therefore y_a = F(a) + c$.

Therefore, $y = F(x) + y_a - F(a)$.

Rearranging the previous equation, the final result is $y(x) = y_a + F(x) - F(a)$.

Example 14

Given $y' = \frac{1}{\sqrt{1-x^2}}$ and $y(0) = 2$, find the solution to the differential equation.

Solution

$$y' = \frac{1}{\sqrt{1-x^2}}$$

$$y = \int \frac{dx}{\sqrt{1-x^2}}$$

$$y = \sin^{-1} x + C$$

$$y(0) = 2 : 2 = \sin^{-1} 0 + C$$

$$C = 2$$

$$\therefore y = \sin^{-1} x + 2$$

Example 15

Find the solution to the following differential equations:

(a) $\frac{dy}{dx} = \sin x$, given $y(0) = 1$ **(b)** $\frac{dy}{dx} = 2 - \cos x$, given $y(0) = 2$.

(c) Hence, use parts **(a)** and **(b)** to solve the second-order differential equation $\frac{d^2y}{dx^2} = \sin x$, given that $\frac{dy}{dx} = 1$ and $y = 2$ where $x = 0$.

Solution

(a) $\frac{dy}{dx} = \sin x$

$y = \int \sin x \, dx$

$y = -\cos x + C$

$y(0) = 1$:

$1 = -1 + C$

$C = 2$

$\therefore \quad y = 2 - \cos x$

(b) $\frac{dy}{dx} = 2 - \cos x$

$y = \int (2 - \cos x) dx$

$y = 2x - \sin x + c$

$y(0) = 2$:

$2 = 2 \times 0 - \sin 0 + c$

$c = 2$

$\therefore \quad y = 2x - \sin x + 2$

(c) Now: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

If $\frac{d^2y}{dx^2} = \sin x$ then the solution of this differential equation will give an equation of the form $\frac{dy}{dx} = f(x)$.

From part **(a)** you can obtain that $\frac{dy}{dx} = 2 - \cos x$.

From part **(b)** you can obtain the solution of this equation, it is $y = 2x - \sin x + 2$.

Hence the solution to $\frac{d^2y}{dx^2} = \sin x$ with the given initial conditions is $y = 2x - \sin x + 2$.