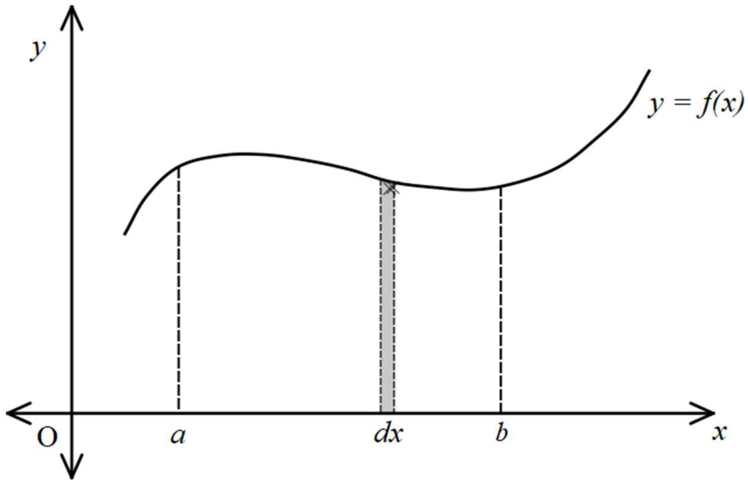
# THE DEFINITE INTEGRAL AND THE AREA UNDER A CURVE

From the examples presented during the previous lesson, it can be seen that as the number of subdivisions increases, the sums of the areas of rectangles become closer to the area under the curve bounded by the two ordinates.




Integral calculus is based on the infinitesimal limit of these rectangular subdivisions.

This process was discovered independently by Leibniz and Newton in the 17th century.

Let $f(x)$ be a continuous function in the interval $a \leq x \leq b$, with $f(x) \geq 0$. Dividing this interval into $n$ subintervals, each small rectangle will have a base $dx$ ($dx$ stands for "small interval of $x$", NOT for "$d$ multiplied by $x$") and a height that is close to $f(x)$ for any value of $x$ in the base. Because $f$ is continuous, all values of $f(x)$ are close together if the values of $x$ are close together. Thus the area of a typical rectangle is $f(x)\,dx$ and the sum of these areas can be written $\sum f(x)\,dx$ [the large Greek letter $\Sigma$ (sigma) denotes "the sum of"].

As $n$ increases (and hence $dx$ decresases), the limiting value of this sum is denoted:

$$\int_a^b f(x)\,dx$$

The symbol $\int$ represents a large elongated S, which stands for 'sum', while the bounds $a$ and $b$ at either end of the $\int$ indicate the interval from $x = a$ to $x = b$.

This $\int_a^b f(x)\,dx$ is called the **definite integral** of the function $f(x)$ between $x = a$ and $x = b$.

The value $A$ of this definite integral is the size of the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

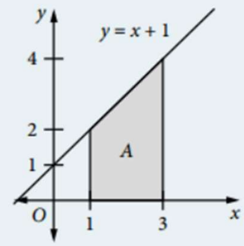
$$\textit{Area under the curve } y = f(x) \textit{ between } x = a \textit{ and } x = b \;=\; \int_a^b f(x)\,dx$$

## Example 3

Write the definite integral for the area of the region under the line $y = x + 1$ between the ordinates $x = 1$ and $x = 3$. By using appropriate area formulae, find the value of this area.

### Solution

Sketch the region: bounds are $f(x) = x + 1$, $x = 1$, $x = 3$, $x$-axis




$$A = \int_a^b f(x)\,dx$$

$$= \int_1^3 (x+1)\,dx$$

The required area is a trapezium. $A = \frac{(2+4)}{2} \times (3-1)$

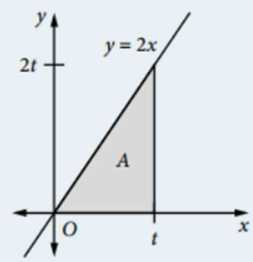
$= 6 \text{ units}^2$

## Example 4

Write the definite integral for the area of the region bounded by the lines $y = 2x$, $x = t$ and the $x$-axis. By using appropriate area formulae, find the value of this area as a function of $t$.

### Solution

Sketch the region: bounds are $f(x) = 2x$, $x = 0$, $x = t$, $x$-axis




$$A = \int_a^b f(x)\,dx$$

$$= \int_0^t 2x\,dx$$

Area of triangle $= \frac{1}{2} \times t \times 2t$

$= t^2$